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O. Neugebauer M. H. Stone O. Veblen

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Mathematical Reviews

Vol. 7, No. 2 FEBRUARY, 1946 Pages 45-104

FOUNDATIONS

Quine, Willard Van Orman. **The Meaning of the New Logic.** Biblioteca de Ciências Sociais. Vol. III. Livraria Martins Editora, São Paulo, 1944. 252 pp. (Portuguese)

[The Portuguese title is "O Sentido da Nova Lógica."] This book is an introduction to symbolic logic based on a course of lectures given by the author at the University of São Paulo in 1942. It is divided into four chapters: (I) the calculus of propositions, (II) the calculus of functions, (III) identity and existence, (IV) class, number, and relation. It is not a formal presentation of any particular system of logic, such as Quine's own system, but a well-rounded résumé of the symbolism and concepts common to modern logical systems. The explanations of logical notions are profusely illustrated by actual examples. The truth-table method is emphasized in the first chapter. Some of the author's own contributions to logic are touched upon. Among the topics treated in the last two chapters are descriptions, the calculus of relations, the avoidance of paradoxes by restrictions placed on the membership relation, the foundations of arithmetic, and the Gödel incompleteness theorem. *O. Frink* (State College, Pa.).

Quine, W. V. On the logic of quantification. J. Symbolic Logic 10, 1–12 (1945).

The author presents a new decision-procedure for the class of "monadic" formulas of the restricted predicate calculus, that is, for formulas which involve only one-place predicate variables. This decision-procedure, which is more convenient to apply than previously known procedures, is established in two steps: first it is shown that every monadic formula can be reduced to a well-determined "basic" formula (that is, to a monadic formula where no quantifier contains within its scope any individual variable besides the variable of quantification); then an easy method, based on truth-tables, is presented for deciding the validity of an arbitrary basic formula. The author calls an arbitrary formula of the restricted predicate calculus monadically valid if it can be obtained by substitution from a valid monadic formula. The decision-procedure of course enables one to decide whether an arbitrary formula is monadically valid. It is shown, furthermore, that every valid formula of the restricted predicate calculus can be derived from monadically valid formulas by application of a generalized rule of modus ponens. This, together with the decision-procedure, suggests a new way of formalizing the restricted predicate calculus. *J. C. C. McKinsey* (Reno, Nev.).

Fitch, Frederic B. A minimum calculus for logic. J. Symbolic Logic 9, 89–94 (1944).

This paper is devoted to the construction of a logic which is basic in the sense of Fitch's previous papers [same J. 7, 105–114 (1942); 9, 57–62 (1944); these Rev. 4, 125; 6, 197]. The present system, however, enjoys the additional property of being minimum in the sense that no weaker basic logic exists (other than a sub-logic of the basic logic). The theorems of such a logic are then logically valid in the sense that they and only they are presupposed in the mere definition of any system. The present theory differs from Fitch's previous system K in certain minor respects, particularly in permitting a direct and simple handling of abstracts of higher degree in terms of abstracts of second degree, and in distinguishing between certain operations not ordinarily distinguished in combinatory logic. It would be of interest to know precisely which of Fitch's results on basic logic apply also mutatis mutandis to the combinatory systems of Curry and Church. *R. M. Martin* (Chicago, Ill.).

Helmer, Olaf, and Oppenheim, Paul. A syntactical definition of probability and of degree of confirmation. J. Symbolic Logic 10, 25–60 (1945).

The authors give a definition of degree of confirmation for the sentences of a language L, of the type of the lower functional calculus, which contains individual constants a_i and a finite number of primitive predicates P_i. First, by means of the p primitive predicates one can form $r=2^p$ "strongest" predicates Q_i, each of which is a product of some of the primitive predicates by the negatives of the rest. It is then shown that, for every ordered set ("distribution") $\Delta=(q_1, \cdots, q_r)$ of nonnegative real numbers such that $q_1+\cdots+q_r=1$, there exists a unique function $p_\Delta(S)$ which is defined for every sentence S of L, which assumes real numbers as values (and, in particular, assumes the value q_i for any sentence S of the form $Q_i a_j$) and which satisfies the customary postulates for probability. The conditional probability $\mathrm{pr}_\Delta(H, E)$ of H with respect to E is then defined, in the usual manner, by the equation

$$\mathrm{pr}_\Delta(H, E)=p_\Delta(H\cdot E)/p_\Delta(E).$$

For a given sentence E of L, the value of $p_\Delta(E)$ will ordinarily depend on Δ. The authors define Δ_E to be the Δ which makes $p_\Delta(E)$ an absolute maximum. The degree of confirmation $\mathrm{dc}(H, E)$ of a hypothesis H on the basis of evidence E is then defined as follows:

$$\mathrm{dc}(H, E)=p_{\Delta_E}(H, E).$$

[In the reviewer's opinion the authors are mistaken in asserting that Δ_E exists for every E. Suppose, for example, that L is a language which contains infinitely many individual constants, but only one primitive predicate P_1, and let E be the sentence '$P_1(a_1)\cdot(\mathrm{Ex})\bar{P}_1(x)$.' Then it is easily seen that for any distribution $\Delta=(q, 1-q)$ we have

$$p_\Delta(E)=q\cdot(1-\lim_{i\to\infty} q^i).$$

Thus $p_\Delta(E)=0$ if $q=0$ or $q=1$, and otherwise $p_\Delta(E)=q$. Hence for every distribution Δ we can find a distribution Δ' such that $p_{\Delta'}(E)>p_\Delta(E)$; therefore $p_\Delta(E)$ has no maximum value, and hence Δ_E does not exist. It follows, of course, that $\mathrm{dc}(H, E)$ does not always exist either, even for noncontradictory E.] *J. C. C. McKinsey* (Reno, Nev.).

Hempel, Carl G., and Oppenheim, Paul. A definition of "degree of confirmation." Philos. Sci. 12, 98–115 (1945). [MF 12794]

This is an exposition in less technical language of the ideas presented in the paper reviewed above.

J. C. C. McKinsey (Reno, Nev.).

Carnap, Rudolf. On inductive logic. Philos. Sci. 12, 72–97 (1945). [MF 12793]

This paper presents a new suggestion for a definition of degree of confirmation. The languages considered are forms of the lower functional calculus, with the sign for identity between individuals, but with only a finite number of constant primitive predicates. Degree of confirmation is defined first for languages with only a finite number of individual constants, and then this definition is extended, by a limiting process, to languages with infinitely many individual constants.

Let L_N be a language with N individual constants $a_1, \cdots, a_N$. By an atomic sentence of L_N is meant a predicate constant of degree r followed by r individual constants. By a state-description is meant a conjunction of some of the atomic sentences of L_N multiplied by the negations of the remaining atomic sentences of L_N. By saying that two state-descriptions are isomorphic, the author means that one can be transformed into the other by a permutation of the individual constants. By the range of a sentence S of L_N is meant the class of those state-descriptions which make S true.

The author introduces a real-valued function $m(S)$, defined for all sentences S of L_N, which satisfies the following conditions: (1) $0 \leqq m(S) \leqq 1$; (2) if S_1 and S_2 are isomorphic state-descriptions, then $m(S_1) = m(S_2)$; (3) if $S_1, \cdots, S_r$ are the totality of state-descriptions isomorphic to the state-description S_1, and $S_1', \cdots, S_s'$ are the totality of state-descriptions isomorphic to the state-description S_1', then $m(S_1 \vee \cdots \vee S_r) = m(S_1' \vee \cdots \vee S_s')$; (4) if S is any sentence of L_N, and if $S_1, \cdots, S_r$ are the state-descriptions in the range of S, then $m(S) = m(S_1) + \cdots + m(S_r)$. This function m obviously satisfies the usual laws of a probability function. In terms of it the author defines the degree of confirmation $c^*(h, e)$ of a hypothesis h on the basis of evidence e as follows: $c^*(h, e) = m(h \cdot e)/m(e)$.

After laying down this definition of confirmation, the author devotes some space to an application of the notion to languages L_N which involve only one-place predicates [thus to languages which are the same, except for involving the identity-sign, as the languages considered in the papers by Helmer and Oppenheim reviewed above]. The mathematical results here are stated without proof. The last few pages of the paper are devoted to a discussion, in terms of the introduced notion of confirmation, of some of the standard forms of inductive inference.

[It is perhaps worth mentioning, in connection with this definition, that the degree of confirmation of a given hypothesis on the basis of given evidence is not independent of the number of primitive predicates occurring in L_N. Thus, for example, if L_N contains the single one-place predicate P_1, then $c^*(P_1(a_2), P_1(a_1)) = 2/3$; but if L_N contains the two one-place predicates P_1 and P_2, then $c^*(P_1(a_2), P_1(a_1)) = 3/5$. (This dependence of $c^*(h, e)$ on the number of predicates in L_N is also evident, for example, from the author's formula (1) on page 86.) It seems difficult to find an intuitive justification for this dependence, and the point would seem worthy of further consideration.] *J. C. C. McKinsey.*

Doss, Raouf. Note on two theorems of Mostowski. J. Symbolic Logic 10, 13–15 (1945).

A theorem due to Mostowski [Fund. Math. 32, 201–252 (1939)] states that the well-ordering theorem cannot be derived from his axioms for set theory (essentially those of Bernays) if we omit the axiom of choice but take the principle of simple-ordering as an axiom. A second theorem of Mostowski [C. R. Soc. Sci. Varsovie, Cl. III. 31, 13–20 (1938); Lindenbaum and Mostowski, ibid., 27–32 (1938)] is to the effect that the principle that every set is either inductive or reflexive is not a consequence of the (Zermelo-Fraenkel) axioms for set theory. The author shows that this principle is not a consequence of Mostowski's axioms, even if the principle of simple-ordering is added as an axiom. It is readily seen that this principle could be derived if the well-ordering theorem were taken as an axiom. Doss's theorem thus contains the two of Mostowski. His proof utilizes fundamentally Mostowski's model. *R. M. Martin.*

Bochvar, D. A. To the question of paradoxes of the mathematical logic and theory of sets. Rec. Math. [Mat. Sbornik] N.S. 15(57), 369–384 (1944). (Russian. English summary) [MF 12303]

Albuquerque, J. On noncontradictory existence. Gaz. Mat., Lisboa 6, no. 25, 4–6 (1945). (Portuguese) [MF 13323]

Potron. Sur les fondements de l'arithmétique. Rev. Gen. Sci. Pures Appl. 51, 141–144 (1940). [MF 12892]

Johnston, Francis E. The postulational treatment of mathematics as exemplified in the theory of groups. American Scientist 33, 39–48, 54 (1945).

Bouligand, Georges. Sur quelques précautions nécessitées par l'exposé des conditions générales propres à l'intervention d'un groupe. Mathematica, Timişoara 20, 90–93 (1944). [MF 12453]

The author first gives an example to show that the familiar but rather vague statement that "those transformations of a set into itself, which leave a given property invariant, form a group" is not true in general. A suitable definition of "property" would seem to be necessary, and the following is suggested. "Let X and Z be two sets and let Ω be a single-valued mapping of X upon Z. Then, for a fixed element z_0 of Z, those elements x of X such that $\Omega(x) = z_0$ are said to have the property P_0." It is pointed out that those transformations of X into itself which leave the property P_0 invariant may not form a group. The question of how a group may be associated with the property P_0 is discussed in the paper reviewed below. *S. A. Jennings.*

Destouches, Jean-Louis. Remarques sur l'association d'un groupe à une propriété au sens de M. G. Bouligand. Mathematica, Timişoara 20, 94–97 (1944). [MF 12454]

Let X be a set and P_0 a "property" as defined in the paper reviewed above. In general there will be many mappings Ω which define the same property P_0, and with every such Ω there is associated the group of transformations of X upon itself which preserve the relationship $\Omega(x) = \Omega(x')$. This group is one of many which leave P_0 invariant. The

author shows that every group associated with an Ω in this way is a subgroup of the causality group belonging to the proposition "x has the property P_0." (The causality group of a proposition p which is either true or false for every element of a set X is the group of transformations of X which leave the truth values of p invariant.)

S. A. Jennings (Vancouver, B. C.).

Court, N. A. Geometry and experience. Scientific Monthly **60**, 63–66 (1945).

Pasturaud, Marie-Thérèse. Le rôle des relations d'équivalence en physique théorique et dans la notion de causalité mathématique. Revue Sci. (Rev. Rose Illus.) **81**, 251–260 (1943). [MF 13804]

NUMBER THEORY

Grossman, Howard D. Applications of an operator to algebra and to number-theory, with comments on the Tarry-Escott problem. Nat. Math. Mag. **19**, 385–390 (1945). [MF 13287]

The problem is to find two sets of numbers with the same sum of kth powers ($k=1, \cdots, m$). The author derives a solution briefly by using a series expansion of the operator D. Comparison is made with various solutions in the literature.

I. Kaplansky (Chicago, Ill.).

Porges, Arthur. A set of eight numbers. Amer. Math. Monthly **52**, 379–382 (1945). [MF 13383]

For x a positive integer, let $G(x)$ denote the sum of the squares of the digits of x (in the denary system). Since $G(x)<x$ for large x, the sequence $\{G^n(x)\}$ either ends in 1 or is ultimately periodic. The author shows that there is in fact one period of length eight: (4, 16, 37, 58, 89, 145, 42, 20). The corresponding problem for cubes is also briefly discussed. Here there appear to be eight distinct periods: (55, 250, 313), (136, 244), (153), (160, 217, 352), (370), (371), (407), (919, 1459). *I. Kaplansky* (Chicago, Ill.).

Williams, G. T. Numbers generated by the function e^{e^x-1}. Amer. Math. Monthly **52**, 323–327 (1945). [MF 12501]

The author proves the following theorems on a set G_n of positive integers defined by

$$eG_n=\sum_{r=0}^{\infty} r^n/r!, \qquad n=0, 1, 2, \cdots.$$

These numbers occur in combinatorial analysis.

$$\text{(I)} \qquad \sum_{r=1}^{\infty}(ar+b)^n/r!=(a\cdot G+b)^{(n)}\cdot e$$

(in this and the following results, G^n is to be interpreted as G_n). (II) $G_{n+1}=(G+1)^{(n)}$. (III) $G_n=G(G-1)^{(n)}$. (IV) Main theorem: $e^{e^x-1}=e^{(G\cdot x)}$.

$$\text{(V)} \qquad G(G-1)\cdots(G-n+1)=1.$$

(VI) $G_p\equiv 2 \pmod p$, p prime (also in the following theorems). (VII) $G_{p+n}\equiv G_n+G_{n+1} \pmod p$.

$$\text{(VIII)} \qquad G_{kp^s+n}\equiv G^{(n)}(G+s)^{(k)} \pmod p.$$

$$\text{(IX)} \qquad G_{n+(p^p-1)/(p-1)}\equiv G_n \pmod p$$

(congruence periods). (X) The sum of $(p^p-1)/(p-1)$ consecutive G's is congruent to 0, modulo p. The methods are elementary. *A. J. Kempner* (Boulder, Colo.).

Brauer, Alfred. A problem of additive number theory and its application in electrical engineering. J. Elisha Mitchell Sci. Soc. **61**, 55–66 (1945). [MF 12987]

Let n be an integer and let $0\leqq a_1<\cdots<a_k\leqq n$ be such that every integer $m\leqq n$ can be expressed in the form a_i-a_j. Denote by $k(n)$ the smallest possible value of k. The author proves that $k(30)=10$. He also discusses the behavior of $k(n)$ for large n. *P. Erdös.*

Racliş, N. Théorème pour les nombres de Fermat. Bull. École Polytech. Bucarest [Bul. Politehn. Bucureşti] **14**, 3–9 (1943). [MF 13571]

The author notes several results like $a^5\equiv a \pmod{30}$. The "theorem" on Fermat numbers $F_n=2^{2^n}+1$ is

$$F_n=2+F_0F_1\cdots F_{n-1}.$$

I. Kaplansky (Chicago, Ill.).

Racliş, N. Démonstration du grand théorème de Fermat pour des grandes valeurs de l'exposant. Bull. École Polytech. Bucarest [Bul. Politechn. Bucureşti] **15**, 3–21 (1944). [MF 13579]

Racliş, N. Lemmes pour le théorème de Fermat. Bull. École Polytech. Bucarest [Bul. Politechn. Bucureşti] **14**, 145–156 (1943). [MF 13576]

The author proves several results, of which the following is typical: if $x^n+y^n=z^n$ in positive integers $x<y<z$, then $n<y$. [That $n<x$ was noted in 1856 by Grunert; cf. Dickson, History of the Theory of Numbers, vol. 2, Washington, 1920, p. 744.] The statement in a summary that he proves $n<y/(1+2^n)$ is not upheld in the text. He hopes to base a proof of Fermat's theorem on the conjecture that, for n prime, $nxyz|(z^n-x^n-y^n)$ if and only if $z=x+y$. The conjecture seems doubtful: for example, for $n=3$, (7, 3, 1) or (39, 7, 2). *I. Kaplansky* (Chicago, Ill.).

Pierre, Charles. Remarques arithmétiques en connexion avec le dernier théorème de Fermat. C. R. Acad. Sci. Paris **218**, 23–25 (1944). [MF 13445]

For a prime $p=6k-1$ the solving of $x^p+y^p+z^p=0$ in integers prime to p is shown to be equivalent to the problem of solving a certain system of congruences (modulo p), whose details are too complicated to reproduce here.

I. Niven (West Lafayette, Ind.).

Vandiver, H. S. New types of relations in finite field theory. II. Proc. Nat. Acad. Sci. U. S. A. **31**, 189–194 (1945). [MF 12796]

As in the first paper by this title [same Proc. **31**, 50–54 (1945); these Rev. **6**, 170] the author obtains congruences involving binomial coefficients by applying his formulas for the number of solutions of equations in finite fields to cases where the number of solutions happens to be known. One such equation is $ax^m=by^m+1$ in a finite field $F(p^{2n})$, p a prime, with certain restrictions on m. As another example the number of solutions (x^e, y^l) of the equation $x^e+by^l+1=0$ in a finite field F of order $p^{2n}=1+cl$ is discussed under certain restrictions. *H. W. Brinkmann* (Swarthmore, Pa.).

Ljunggren, Wilhelm. Über die Gleichung $x^4-Dy^2=1$. Arch. Math. Naturvid. **45**, no. 5, 61–70 (1942). [MF 12972]

It is proved that for any positive integer D the equation $x^4-Dy^2=1$ has at most two solutions in positive integers x and y. [The author imposes the further hypothesis that

D is square-free, which seems unnecessary.] The proof, which furnishes a procedure for calculating the possible solutions, is based on earlier work of the author [Skr. Norske Vid. Akad. Oslo. I, 1936, no. 12 (1937)] on the units of quadratic and biquadratic fields. The result is the best possible: the equation with $D=1785$ has the solutions $x=13$, $y=4$ and $x=239$, $y=1352$. *I. Niven.*

Selberg, Sigmund. Ein elementarer Satz über den grössten Primfaktor einer quadratfreien Zahl, die aus einer gegebenen Anzahl von Primfaktoren zusammengesetzt ist. Arch. Math. Naturvid. 45, no. 4, 53–60 (1942). [MF 12971]

Denote by $a_1<\cdots<a_z\leqq n$ the integers having exactly k prime factors. It is well known that

$$z=(1+o(1))(n/\log n)(\log\log n)^{k-1}/(k-1)!.$$

Denote by $p(a_i)$ the greatest prime factor of a_i. Let c be arbitrary, $0<c<1$. The author proves that $p(a_i)>n^c$ for all but

$$o\left(\frac{n}{\log n}\cdot\frac{(\log\log n)^{k-1}}{(k-1)!}\right)$$

values of n. [A sharper result obtained by the author has already been reviewed [Skr. Norske Vid. Akad. Oslo. I. 1942, no. 5 (1942); these Rev. 6, 57].] *P. Erdös.*

Thébault, Victor. Sur les nombres premiers impairs. C. R. Acad. Sci. Paris 218, 223–224 (1944). [MF 13389]

If n odd primes form an arithmetic progression then the common difference is divisible by the product of all primes less than n. If n is a prime but not a term of the progression then the common difference is also divisible by n.

H. S. Zuckerman (Seattle, Wash.).

Bouligand, Georges. Notions sur la répartition des nombres premiers. Rev. Sci. (Rev. Rose Illus.) 78, 333–344 (1940). [MF 13315]

Summary of two expository lectures.

Brun, Viggo. The study of the prime numbers from antiquity to our time. Norske Vid. Selsk. Forh. 15, 16 pp. (1942). (Norwegian) [MF 13360]

Contains a bibliography of 71 papers on additive number theory.

Selberg, Atle. On the normal density of primes in small intervals, and the difference between consecutive primes. Arch. Math. Naturvid. 47, no. 6, 87–105 (1943). [MF 12976]

The relation

$$\pi(x+\Phi(x))-\pi(x)\sim\Phi(x)/\log x,\qquad x\to\infty, \tag{1}$$

where $\Phi(x)$ is a positive increasing function of x such that $\Phi(x)/x$ is decreasing, was shown by A. E. Ingham [Quart. J. Math., Oxford Ser. 8, 255–266 (1937)] to hold if

$$\liminf_{x\to\infty}\frac{\log\Phi(x)}{\log x}>\frac{48}{77}. \tag{2}$$

On the Riemann hypothesis, (1) holds provided

$$\Phi(x)/(x^{\frac12}\log x)\to\infty,\qquad x\to\infty, \tag{3}$$

as may be deduced by arguments of H. Cramér [Ark. Mat. Astr. Fys. 15, no. 5 (1921)].

The author shows that, on the Riemann hypothesis, (1) holds for almost all x provided the weaker condition

$$\Phi(x)/(\log x)^2\to\infty,\qquad x\to\infty, \tag{4}$$

is satisfied by $\Phi(x)$. By "for almost all values of x" is meant: as $x\to\infty$ through any sequence of real numbers lying outside a certain set S of mean density zero in $(0,\infty)$. This is near to a best possible result, since the conclusion becomes false on replacing (4) by any condition weaker than $\Phi(x)/\log x\to\infty$.

Further theorems improve on two results of H. Cramér [Acta Arith. 2, 23–46 (1936)] concerning the gaps in the prime number sequence by showing, still on the Riemann hypothesis, that the sum of p_n-p_{n-1} over the range $p_n\leqq x$, $p_n-p_{n-1}\geqq(H/x)p_n$, is $O((x/H)\log^2 x)$, where $H=x^\alpha\log^\beta x$, $0\leqq\alpha\leqq1$, $\beta>0$, and that

$$\sum_{p_n\leqq x}(p_n-p_{n-1})^2/p_n=O(\log^3 x).$$

Finally, it is shown independently of the Riemann hypothesis that (1) holds for almost all x when 48 is replaced by 19 in (2). *E. H. Linfoot* (Bristol).

Selberg, Atle. On the zeros of Riemann's zeta-function on the critical line. Arch. Math. Naturvid. 45, no. 9, 101–114 (1942). [MF 12982]

The result $N_0(2T)-N_0(T)>KT\log\log\log T$ is proved, where $N_0(T)$ is the number of zeros of $\zeta(s)$ with $\sigma=\frac12$, $0<t<T$. The factor $\log\log\log T$ represents an improvement over the result of Hardy and Littlewood. Certain other possible improvements are mentioned without proofs.

H. S. Zuckerman (Seattle, Wash.).

a later & stronger result has already been reviewed; cf. these Rev. 6, 58.

Taylor, P. R. On the Riemann zeta function. Quart. J. Math., Oxford Ser. 16, 1–21 (1945). [MF 12787]

A number of disconnected results collected by J. E. Reese and E. C. Titchmarsh from the author's work. One result is that if $\xi_1(s)=\pi^{-\frac12 s}\Gamma(\frac12 s)\zeta(s)$ then $\xi_1(s+\frac12)-\xi_1(s-\frac12)$ has all its complex zeros on $\sigma=\frac12$. Some of the other results are attempts to supplement this with the proof of Riemann's hypothesis in view. *H. S. Zuckerman* (Seattle, Wash.).

Haviland, E. K. On the asymptotic behavior of the Riemann ξ-function. Amer. J. Math. 67, 411–416 (1945). [MF 12921]

If $\zeta(s)$ is the Riemann zeta-function,

$$\xi(s)=\tfrac12 s(s-1)\pi^{-\frac12 s}\Gamma(\tfrac12 s)\zeta(s),$$

and $M(r)$ is the maximum of $|\xi(\frac12+is)|$ on the circle $|s|=r$, then

$$M(r)\sim(\tfrac12\pi)^{\frac14}(2\pi e)^{-\frac12 r}r^{\frac12 r+7/4}$$

as $r\to\infty$. This is the first term of an infinite asymptotic series. *E. C. Titchmarsh* (Oxford).

Selmer, Ernst S. Eine neue hypothetische Formel für die Anzahl der Goldbachschen Spaltungen einer geraden Zahl, und eine numerische Kontrolle. Arch. Math. Naturvid. 46, no. 1, 1–18 (1943). [MF 12977]

The author uses $G(2n)$ to denote the number of decompositions (Goldbach decompositions) of $2n$ as the sum of two primes, where $2n=p+q$ and $2n=q+p$ are considered the same decomposition. He derives the following result:

$$H(2x)\sim\tfrac12\int_2^{2x-2}\frac{\mathrm{li}\,(2x-u)}{\log u}\,du,$$

where $H(2x)=\sum_{v=1}^{x}G(2v)$, li (x) is the Cauchy principal value integral $\int_0^x ds/\log s$ and $f(x)\sim g(x)$ means that $f(x)/g(x)$

approaches 1 as x becomes infinite. A consequence is that

$$dH(2n)/dn \sim 2\int_0^n \frac{dx}{\log(n+x)\cdot\log(n-x)}.$$

Computations are recorded confirming this last result very closely for the 51 even numbers from 1,000,000 to 1,000,100.

B. W. Jones (Ithaca, N. Y.).

Selmer, Ernst S. Eine numerische Untersuchung über die Darstellung der natürlichen Zahlen als Summe einer Primzahl und einer Quadratzahl. Arch. Math. Naturvid. 47, no. 2, 21–39 (1943). [MF 12973]

The author denotes by $R(n)$ the number of "Romanoff decompositions" of n, that is, the number of ways in which it can be expressed as the sum of a prime number and a square. He cites the empirical result of Hardy and Littlewood, $R(n)\sim wn^{\frac{1}{2}}/\log n$, where

$$w=\prod_{p=3}^{\infty}\left\{1-\frac{1}{p-1}\left(\frac{n}{p}\right)\right\}$$

and $(\frac{n}{p})$ is the Legendre symbol. He proves

$$R_m(n)\sim\int_0^{n^{\frac{1}{2}}}\frac{dx}{\log(n-x^2)},$$

where $R_m(x)$ is the derivative with respect to x of $\sum_{v=1}^{x}R(v)$, and also

$$R_m(n)\approx\frac{n^{\frac{1}{2}}}{\log n-2(1-\log 2)}w.$$

He gives a table of the number of Romanoff decompositions of the numbers from 1 to 2000 and from 1,010,000 to 1,010,100, verifying the above results.

B. W. Jones (Ithaca, N. Y.).

Siegel, Carl Ludwig. Sums of mth powers of algebraic integers. Ann. of Math. (2) 46, 313–339 (1945). [MF 12405]

Let K be an algebraic number-field of degree n with r real conjugates $K^{(1)},\cdots,K^{(r)}$. An integer ν of K is totally positive if $\nu^{(1)},\cdots,\nu^{(r)}$ are all positive. This paper deals with the representability of all totally positive integers of K as sums of integral squares in K, or more generally as sums of totally positive integral mth powers in K. The main theorems are as follows. (1) If K is totally real, representability by squares holds if and only if K is either the rational field R or the field $R(\sqrt{5})$. (2) If K is not totally real, representability by squares holds if and only if the discriminant of K is odd, and then five squares suffice. The author conjectures that four squares may suffice. (3) If K is totally real and $m>2$, representability by mth powers holds if and only if $K=R$. (4) If K is not totally real, representability by mth powers holds if and only if every integer in K is representable as $\pm\lambda_1^m\pm\lambda_2^m\pm\cdots$ with integral $\lambda_1,\lambda_2,\cdots$ in K. If this condition is satisfied, $(2^{m-1}+n)mn+1$ terms suffice. The proofs of (1) and (3) are "elementary." The proofs of (2) and (4) are by further extensions of the author's generalization of the Hardy-Littlewood method to algebraic number-fields. Naturally these proofs are of considerable complexity, and they involve a knowledge of two of the author's earlier papers on this generalization [Math. Ann. 88, 184–210 (1923); Amer. J. Math. 66, 122–136 (1944); these Rev. 5, 200].

H. Davenport (London).

Bell, E. T. Universal rational functions. Proc. Nat. Acad. Sci. U. S. A. 31, 317–319 (1945). [MF 13440]

A rational function F/G, where F and G are polynomials in n variables with positive integer coefficients, is here called universal for C, a subset of the set of all positive integers, if all integers in C are represented by F/G for positive integer values of the variables. Examples of such functions are given and some properties indicated. *B. W. Jones.*

Pillai, S. S. Highly composite numbers of the tth order. J. Indian Math. Soc. (N.S.) 8, 61–74 (1944). [MF 13269]

The author generalizes Ramanujan's concept of highly composite numbers [cf. Alaoglu and Erdös, Trans. Amer. Math. Soc. 56, 448–469 (1944); these Rev. 6, 117] by letting $d_t(N)$ denote the number of ways of decomposing N into t factors and by calling N highly composite of the tth order if $d_t(N)>d_t(N')$ for all $N'<N$. He shows that such an N can be written in the form $2^{a_2}3^{a_3}\cdots p^{a_p}$, where $a_2\geqq\cdots\geqq a_p\geqq 1$; that $a_p=1$ for all except a finite number of N; that

$$a_\lambda\log\lambda=(t-1)\log p/\log t+O(\sqrt{(\log p\log\log p)});$$

and he gives various results about the distribution of equal consecutive indices a_i. The notion of superior highly composite numbers of the tth order is developed.

B. W. Jones (Ithaca, N. Y.).

Siez, Ching-Syur. A general expression for Euler's ϕ-function. J. Indian Math. Soc. (N.S.) 8, 91–94 (1944). [MF 13272]

Put $\mu_k(p^l)=0$ or $(-1)^l\binom{k}{l}$ according as $l>k$ or $l\leqq k$, and $\mu_k(n)=\prod\mu_k(p^l)$, where $n=\prod p^l$; put

$$\sigma_k(n)=n\prod_{p|n}\{1+\tbinom{k-1}{1}p^{-1}+\tbinom{k}{2}p^{-2}+\cdots+\tbinom{k+l-2}{l}p^{-l}\}.$$

The author proves the formula

$$\phi(n)=\sum_{d|n}\mu_k(n/d)\sigma_k(d).$$

L. Carlitz (Durham, N. C.).

Erdös, Paul. Some remarks on Euler's ϕ-function and some related problems. Bull. Amer. Math. Soc. 51, 540–544 (1945). [MF 12815]

Let $f(n)$ be the number of integers $m\leqq n$ for which $\phi(x)=m$ has solutions, $\phi(x)$ being Euler's ϕ-function. The author proves that $f(n)>cn(\log n)^{-1}\log\log n$, for some constant c. In an earlier paper he had proved that $f(n)=o(n(\log n)^{1-\epsilon})$ for every $\epsilon>0$ [Quart. J. Math., Oxford Ser. 6, 205–213 (1935)]. The sharper result $f(n)>n(\log n)^{-1}(\log\log n)^k$, for every k, is stated but not proved. It is also proved that the number of integers $m\leqq n$ for which $\phi(m)\leqq n$ is $cn+o(n)$; this result is due to the author and Turán. It is stated that exactly parallel results can be proved if $\phi(m)$ is replaced by $\sigma(m)$, the sum of the divisors of m. *H. W. Brinkmann.*

Sathe, L. G. On a congruence property of the divisor function. Amer. J. Math. 67, 397–406 (1945). [MF 12919]

Let $N(k,r,x)$ denote the number of $n\leqq x$ for which $d(n)\equiv r\pmod k$, $d(n)$ being the number of divisors of n. The author proves that, if 2^m is the highest power of 2 which divides k, then there is a constant $B=B(k,r)>0$ such that $N(k,r,x)\sim Bx$ if $2^m|r$, and $N(k,r,x)=o(x)$ otherwise. For k prime and $r=0$ this result was given by Pillai [J. Indian Math. Soc. (N.S.) 6, 118–119 (1942); these Rev. 4, 210], whose work contained an error corrected by Sathe [J. Indian Math. Soc. (N.S.) 7, 143–145 (1943); these Rev.

6, 36]. Two miscellaneous results are obtained: it is shown that $B(k, r) = B(k', r')$ under certain conditions on k, r, k', r'; and $B(k, r)$ is evaluated in terms of the zeta function in case $r=0$ and $k=2^m p$, p a prime. *I. Niven.*

Gupta, Hansraj. Congruence properties of $\tau(n)$. Proc. Benares Math. Soc. **5**, 17–22 (1943). [MF 12533]

Ramanujan's function $\tau(n)$ is generated by

$$\sum_{n=1}^{\infty} \tau(n) x^n = x \prod_{m=1}^{\infty} (1-x^m)^{24}.$$

A proof is given that, for odd n, $\tau(n)$ is congruent modulo 8 to the number of solutions of the equation $4n = a^2+b^2+c^2+d^2$, where a, b, c, d are odd and positive. Separate proofs are given for the special cases $\tau(p) \equiv 2 \pmod 4$ if $p \equiv 1 \pmod 4$ is a prime; $\tau(p) \equiv 4 \pmod 8$ if $p \equiv 3 \pmod 8$ is a prime; $\tau(8k+7) \equiv 0 \pmod 8$. The theorem, in view of a result of Legendre, could have been stated as follows:

$$\tau(2m+1) \equiv \sigma(2m+1) \pmod 8,$$

where $\sigma(k)$ is the sum of the divisors of k, from which the special cases follow at once [cf. Ramanathan, Proc. Indian Acad. Sci., Sect. A. **19**, 146–148 (1944); Math. Student **11**, 33–35 (1943); these Rev. **6**, 37]. The author proves that $\tau(n)$ is odd if and only if n is an odd square. On page 22 there is a table of $\tau(n)$ for $n \leqq 130$. A comparison of this table with an earlier more extensive table [Lehmer, Duke Math. J. **10**, 483–492 (1943); these Rev. **5**, 35] reveals no discrepancies. *D. H. Lehmer.*

Banerjee, D. P. On the application of the congruence property of Ramanujan's function to certain quaternary form. Bull. Calcutta Math. Soc. **37**, 24–26 (1945). [MF 13305]

The author considers the number $N_k(n)$ of solutions $(r_1, \cdots, r_k)$ of

$$kn = \sum_{i=1}^{k} (2r_i+1)^2, \qquad k=2, 4, 8,$$

where n is a given odd integer, and notes that

$$N_4(n) \equiv N_8(n) \equiv \tau(n) \pmod 2,$$
$$N_2(n) \equiv \tau(n) \pmod 4,$$

where $\tau(n)$ is Ramanujan's function. [The author's remarks on the number of solutions of

$$4n = a_1^2 + a_2^2 + a_3^2 + a_4^2 + du^2$$

appear to be erroneous.] *D. H. Lehmer.*

Ramanathan, K. G. On Ramanujan's trigonometrical sum $C_m(n)$. J. Madras Univ. Sect. B. **15**, 1–9 (1943). [MF 12316]

This paper is concerned with a sum which is, in fact, the sum of the nth powers of the primitive mth roots of unity. The author points out its connection with partitions modulo m, the sum in question being the excess of the number of partitions of n into an even number of incongruent parts modulo m over those into an odd number of such parts. Simple proofs are given of a number of known theorems, such as the one that asserts that the product of 2 sin $(\pi n/m)$ taken over all n less than and prime to m has the value p or 1 according as m is or is not a power of the prime p.

D. H. Lehmer (Aberdeen Proving Ground, Md.).

Kostandi, G. Une propriété des équations irréductibles de la division du cercle. Bull. École Polytech. Bucarest [Bul. Politehn. București] **14**, 10–18 (1943). [MF 13572]

This paper is concerned with the magnitude of the coefficients of the irreducible cyclotomic polynomial $Q_n(x)$ whose roots are the primitive nth roots of unity. All the results obtained are known. A proof is given that the coefficients have values ± 1 or 0 whenever $n = 2^\alpha p^\beta q^\gamma$ ($\alpha \geqq 0, \beta \geqq 0, \gamma \geqq 0$), where p and q are odd primes. The paper concludes with a proof that there exist integers n for which certain of the coefficients of $Q_n(x)$ are greater in absolute value than any preassigned number. The proof is identical with that of I. Schur [see E. Lehmer, Bull. Amer. Math. Soc. **42**, 389–390 (1936)]. *D. H. Lehmer.*

Jones, Burton W. A canonical quadratic form for the ring of 2-adic integers. Duke Math. J. **11**, 715–727 (1944). [MF 11574]

The author seeks an easily applicable criterion for rational equivalence of two quadratic forms. He observes that, since Siegel [Amer. J. Math. **63**, 658–680 (1941); these Rev. **3**, 163] has shown that two integral forms belong to the same genus if and only if they are equivalent in the real field and in every ring R_p of p-adic integers, it suffices to study equivalence of forms in R_p. The major part of the paper is devoted to the difficult case $p=2$. The author obtains a unique "canonical" representative for every set of forms equivalent in R_2. He shows that every symmetric matrix in R_2 is equivalent in R_2 to a matrix (1) $\mathfrak{A} = \{2^{t_1}\mathfrak{A}_1, \cdots, 2^{t_s}\mathfrak{A}_s\}$, where, for every i, $t_i < t_{i+1}$ and $|\mathfrak{A}_i|$ is a unit. Furthermore, every $\mathfrak{A}_i$ is either in the form $\sum_{k=1}^{n_i} a_k x_k^2$, where a_k are prescribed odd residues modulo 8, or in one of the forms $\{\mathfrak{T}, \cdots, \mathfrak{T}\}$ or $\{\mathfrak{T}, \cdots, \mathfrak{T}, \mathfrak{S}\}$, where $\mathfrak{T} = \binom{0\ 1}{1\ 0}$, $\mathfrak{S} = \binom{2\ 1}{1\ 2}$ and the following conditions are fulfilled. (a) If $\mathfrak{A}_{i+1}$ is properly primitive and $t_{i+1} = t_i + 1$, then $\mathfrak{A}_i = \{\mathfrak{T}, \cdots, \mathfrak{T}\}$ if $\mathfrak{A}_i$ is improperly primitive, while $\lambda(\mathfrak{A}_i) = 1 \equiv |\mathfrak{A}_i| \pmod 4$ if $\mathfrak{A}_i$ is properly primitive. (b) If $\mathfrak{A}_i$ is properly primitive and $x'\{2^{t_{i+1}-t_i}\mathfrak{A}_{i+1}, 2^{t_{i+2}-t_i}\mathfrak{A}_{i+2}\} \equiv 4 \pmod 8$ solvable, then $|\mathfrak{A}_i| \equiv \pm 1 \pmod 8$. (c) If the initial conditions for both (a) and (b) hold with $\mathfrak{A}_i$ properly primitive, then $|\mathfrak{A}_i| \equiv 1 \pmod 8$ and $\lambda(\mathfrak{A}_i) = 1$. Here $\lambda(\mathfrak{A}) = -c_2(\mathfrak{A})(-1 \,|\, |\mathfrak{A}|)$, where c_2 is Hasse's invariant. In view of the uniqueness of (1), the problem of rational equivalence of two forms is reduced to the determination of their canonical forms.

A. E. Ross (St. Louis, Mo.).

Pall, Gordon. The arithmetical invariants of quadratic forms. Bull. Amer. Math. Soc. **51**, 185–197 (1945). [MF 12088]

This lecture is a survey of the recent developments in the study of the arithmetic invariants of quadratic forms with particular emphasis placed on the author's own contributions to the subject; a more detailed account is to appear in a forthcoming book. The major aim is the construction of a complete system of generic invariants which can be readily evaluated with the help of an easily computable canonical form. It is shown that, if p is a finite prime, then every integral $n$ary quadratic form f can be expressed (by means of a rational transformation which is integral modulo p and has determinant $+1$) in the form

$$p^{e_1}\phi_1 + \cdots + p^{e_s}\phi_s, \qquad e_1 < \cdots < e_s,$$

where each ϕ_i denotes a form with integral coefficients modulo p, the variables in different ϕ_i do not overlap, $|\phi_i|$ is prime to p ($i=1, \cdots, s$) and the e_i are integers. Also,

$e_1 \geqq 0$, except that $e_1 = -1$ if $p=2$ and f has any odd cross-product coefficient. If n_i denotes the number of variables in ϕ_i, $\sum n_i = n$ and $\sum e_i n_i$ is the exponent to which p appears in f. The proof (which serves also as the method of reduction) amounts to little more than judiciously completing squares. A complete system of invariants is then defined in terms of the above canonical form and their use is illustrated by an example. A very neat and compact condition for the existence of a quadratic form in n variables with prescribed invariants is given. It consists of the equality $\prod_p c_p(f) = +1$ and the assumption that the product of all the characters modulo $p > 2$ has the same value as $(d_1|p)$, where $d = p^a d_1$, $(d_1, p) = 1$. Here $c_p(f)$ are generic invariants of f.

The author points out that although the ideas and methods employed are essentially those of Hensel [Zahlentheorie, Göschen, Berlin-Leipzig, 1913] and Hasse [J. Reine Angew. Math. **152**, 129–148, 205–224 (1923)] there are three aspects in which his treatment differs from theirs. First, in defining generic invariants he introduces a symbol which, although essentially equal to Hilbert's norm-residue symbol, is defined more simply and symmetrically. Second, he does not use p-adic numbers as such, but replaces them by rational congruences. Third, in deriving all of the results of this article, the author avoids the use of the deep result of Dirichlet on the existence of primes in an arithmetical progression and employs instead the essentially more elementary theorem of Gauss on the existence of a genus of binary quadratic forms. As an illustration of this approach he gives a very simple proof of Legendre's theorem that every positive integer not of the form $4^k(8n+7)$ is a sum of three squares. *A. E. Ross* (St. Louis, Mo.).

Siegel, Carl Ludwig. The average measure of quadratic forms with given determinant and signature. Ann. of Math. (2) **45**, 667–685 (1944). [MF 11387]

In the first part of this paper the author proves (in two ways) that

$$(1)\qquad \sum_{k \leqq N} h_{4k} \log \epsilon_{4k} \sim 4\pi^2 N^{\frac{3}{2}}/(21\zeta(3)), \qquad N\to\infty.$$

This result is equivalent to a hitherto unproved assertion of Gauss that the mean value of the expression $h_{4k} \log \epsilon_{4k}$ is asymptotically equal to $2\pi^2 k^{\frac{1}{2}}/(7\zeta(3))$. Here h_d is the number of classes of primitive binary quadratic forms $Q(x, y) = ax^2 + bxy + cy^2$ of positive discriminant $d = b^2 - 4ac$; two forms belong to the same class if they can be transformed into one another by an integral transformation of determinant $+1$; if d is not a perfect square, $\epsilon_d = \frac{1}{2}(t + u\sqrt{d})$, where t, u are the smallest positive integral solutions of $t^2 - du^2 = 4$.

The author observes that both methods of proof can be used to determine the average of $h_d \log \epsilon_d$ in any class of residues $d \equiv d_0 \pmod m$. The first proof contains explicitly the results for $m=4$ and $d_0 = 0, 1$ and yields the formula

$$(2)\qquad \sum_{k \leqq N} h_k \log \epsilon_k = \pi^2 N^{\frac{3}{2}}/(18\zeta(3)) + O(N \log N).$$

It is pointed out that the first method applied to positive binary forms leads to the corresponding results

$$(1')\qquad \sum_{k \leqq N} h_{-4k} \sim 4\pi N^{\frac{3}{2}}/(21\zeta(3)),$$

$$(2')\qquad \sum_{k \leqq N} h_{-k} = \pi N^{\frac{3}{2}}/(18\zeta(3)) + O(N \log N).$$

Mertens [J. Reine Angew. Math. **77**, 289–339 (1874)] proved (1′) by direct computation of the lattice points in the three-dimensional domain $|b| < a < c$, $0 < 4ac - b^2 < T$. Application of a corresponding idea in the case of indefinite forms serves as the basis for the second proof of (1).

The second part of the paper contains a geometric treatment of the analogous problem for quadratic forms $\mathfrak{S}[\mathfrak{x}]$ in m variables. It is shown that

$$(3)\qquad \sum_{S \leqq N} S\rho(\mathfrak{S}) \sim \tfrac{1}{2} N \prod_{k=2}^{m} \zeta(k), \qquad N\to\infty.$$

Here the summation is extended over a system of representatives $\mathfrak{S}$ of all classes of given signature $n, m-n$ whose determinants have the absolute value $S \leqq N$, and $\rho(\mathfrak{S})$ is defined as follows: let $\mathfrak{T}$ be a variable real symmetric matrix of signature $n, m-n$ which lies in a domain T, let Y be the domain in the space of all real m-rowed matrices $\mathfrak{Y}$ which is mapped into T by the condition $\mathfrak{S}[\mathfrak{Y}] = \mathfrak{T}$, and let Y_0 be a fundamental domain in Y with respect to the group of units of $\mathfrak{S}$. Then, if $v(Y_0)$ and $v(T)$ are the Euclidean volumes of Y_0 and T, where the m^2 elements of $\mathfrak{Y}$ and the $\frac{1}{2}m(m+1)$ independent elements of $\mathfrak{T}$ are taken as rectangular Cartesian coordinates, we have

$$\rho(\mathfrak{S}) = \lim_{T\to\mathfrak{S}} v(Y_0)/v(T)$$

if the limit exists, and $\rho(\mathfrak{S}) = 0$ otherwise. It is brought out by the author that in case $n=0$ or $n=m$ (3) follows from a result of Minkowski [J. Reine Angew. Math. **129**, 220–274 (1906)]. *A. E. Ross* (St. Louis, Mo.).

Hua, Loo-Keng. A remark on a result due to Blichfeldt. Bull. Amer. Math. Soc. **51**, 537–539 (1945). [MF 12814]

Let $\xi_1, \cdots, \xi_n$ be n linear forms in $x_1, \cdots, x_n$ of unit determinant, of which $n-2s$ are real and $2s$ are complex conjugate in pairs ($n \geqq 3$, $2s \leqq n$). The convex body

$$F(x_1, \cdots, x_n) \equiv |\xi_1|^\sigma + \cdots + |\xi_n|^\sigma \leqq 1, \qquad \sigma \geqq 1,$$

is of volume

$$V(\sigma) = 2^{n-(1+2/\sigma)s}\pi^s \Gamma^{n+2s}(1+1/\sigma)\Gamma^s(1+2/\sigma)\Gamma^{-1}(1+n/\sigma),$$

and so, by Minkowski's theorem [Geometrie der Zahlen, Leipzig, 1910, p. 76], there are integers $(x_1, \cdots, x_n) \neq (0, \cdots, 0)$ such that $F(x_1, \cdots, x_n) \leqq r^\sigma$ if $r^n \geqq 2^n V(\sigma)^{-1}$. Van der Corput and Schaake [Acta Arith. **2**, 152–160 (1936)] improved this, by means of Blichfeldt's method [Trans. Amer. Math. Soc. **15**, 227–235 (1914)], to

$$r^n \geqq \epsilon(\sigma)^{n/\sigma}(n+\sigma)\sigma^{-1}V(\sigma)^{-1},$$

where $\epsilon(\sigma)$ is the smallest constant such that, for all k and all complex $z_1, \cdots, z_n$,

$$\sum_{p,q=1}^{k} |z_p - z_q|^\sigma \leqq \epsilon(\sigma) k \sum_{p=1}^{k} |z_p|^\sigma.$$

They also proved that $\epsilon(\sigma) = 2^{\sigma-1}$ if $\sigma \geqq 2$. The author deduces this result in a few lines from M. Riesz's convexity theorem [Hardy, Littlewood and Pólya, Inequalities, Cambridge University Press, 1934, p. 219, theorem 296]. He also shows that $\epsilon(\sigma) = 2$ if $1 \leqq \sigma \leqq 2$, and gives another slight improvement if $1 \leqq \sigma \leqq 1.865 \cdots$. *K. Mahler.*

Ollerenshaw, Kathleen. The minima of a pair of indefinite, harmonic, binary quadratic forms. Proc. Cambridge Philos. Soc. **41**, 77–96 (1945). [MF 12839]

Let K_μ be the set of points (x, y) in the Euclidean plane for which $|xy| \leqq 1$, $|x^2 - y^2| \leqq 2\mu$. A K_μ-admissible lattice Λ is defined to be a lattice such that the origin is its only point interior to K_μ. Let $d(\Lambda)$ denote the area of a fundamental parallelogram of Λ. The lower bound of $d(\Lambda)$ for all

K_μ-admissible lattices is designated by $\Delta(K_\mu)$. A lattice for which $d(\Lambda)=\Delta(K_\mu)$ is known as a critical lattice. In this paper $\Delta(K_\mu)$ is determined for all positive μ and all critical lattices are constructed. Except for the use of a result of Mahler, the methods are elementary.

By means of an affine transformation of these results the author obtains the principal result of the paper, which is as follows. Let $f_1(x, y)$, $f_2(x, y)$ be a pair of real, binary, indefinite quadratic forms $a_1x^2+2b_1xy+c_1y^2$, $a_2x^2+2b_2xy+c_2y^2$ for which the harmonic relation $a_1c_2-2b_1b_2+c_1a_2=0$ holds. Then the minimum number k is determined, depending only on the discriminants d_1 and d_2 of the forms, for which integers x, y exist, not both zero, so that $|f_1(x, y)|\leqq k$, $|f_2(x, y)|\leqq k$. When the equality sign holds in at least one of the above relations the pair of forms is defined to be critical. A critical pair is constructed for each pair of discriminant values d_1 and d_2. *D. Derry* (Saskatoon, Sask.).

Segre, B. On arithmetical properties of singular cubic surfaces. J. London Math. Soc. 19, 84–91 (1944). [MF 12554]

The question considered is the existence of points with rational coordinates on a cubic surface $F(x, y, z)=0$ with rational coefficients and one or more singular points. The answer depends on the configuration of the singular points. The only remaining nontrivial cases are those of two or three isolated singular points. [The four point case has been considered by the author in another paper [same J. 19, 46–55 (1944); these Rev. 6, 117].] For three singular points there is either no rational point or a two parameter set, both cases being possible. For two singularities there is an infinite number of rational points but not a two parameter set; proof of this last fact is not given.
R. J. Walker (Ithaca, N. Y.)

Puri, Amritsagar. Transcendence of decimals. Math. Student 12, 88–90 (1945). [MF 12491]

Lacuna theorems on the algebraic or transcendental character of numbers represented by series. (I) If the sequence $q_{n+1}-rq_n$ is unbounded,

$$x=a_0+a_{q_1}/k^{q_1}+a_{q_2}/k^{q_2}+\cdots$$

cannot satisfy an algebraic equation with integral rational coefficients of degree less than $r+1$. Here a_i, q_i are integers, $a_0\geqq 0$, $a_{q_n}>0$, $q_n<q_{n+1}$ and k is an integer not less than 2. In particular, if the sequence q_{n+1}/q_n is unbounded, x is transcendental. (II) If (a) the positive integer u_n is divisible by the least common multiple of $u_1, u_2, \cdots, u_{n-1}$ and (b) $2^{k_n}\leqq u_n\leqq 2^{k_n+1}$, then $\sum_1^\infty a_n/u_n$ (a_i positive integers) is transcendental if the sequence k_{n+1}/k_n is unbounded, and does not satisfy an algebraic equation with rational coefficients of degree less than $r+1$ if the sequence $k_{n+1}-rk_n$ is unbounded. The proofs are elementary. [In formula (3.4), read ξ for ξ_r; in theorem 3 read: "if the sequence $p_{n+1}-rp_n$ is unbounded."] *A. J. Kempner* (Boulder, Colo.).

ANALYSIS

Calculus

★Ryžik, I. M. Tablicy Integralov, Summ, Ryadov i Proizvedenii [Tables of Integrals, Sums, Series and Products]. OGIZ, Moscow-Leningrad, 1943. 400 pp. (Russian)

The first 80 pages are devoted to indefinite integrals involving elementary functions, the next 31 to elliptic integrals and the next 127 to definite integrals, a few of which are multiple integrals or contour integrals. There are 98 pages of series which can be summed either in terms of elementary functions or by means of the higher functions of analysis. Nearly 50 pages are devoted to a summary of methods of numerical integration, the transformation of multiple integrals, methods of summation and Fourier's theorem. Results involving Bessel functions are included. The notation used is similar to that used by Whittaker and Watson, except that $\Gamma(y+z-1)/\{\Gamma(y)\Gamma(z)\}$ is denoted by $s(y, z)$. Tables with 10 decimal places are given at the end for 15 values of $2^{-2n}s(n+1, n+1)$ and of this quantity divided, respectively, by $2n+1$, by $2n-1$ and by $2s(2n+2, 3)$. Series of Legendre functions and series that can be summed by means of hypergeometric functions add to the value of the book. There are also tables of Stirling numbers and of the coefficients needed for the Gaussian method of quadrature. *H. Bateman* (Pasadena, Calif.).

Simpson, Harold. A certain multiple integral. Philos. Mag. (7) 36, 224 (1945). [MF 13349]

Comment on a paper by A. S. Gladwin [Philos. Mag. (7) 35, 657–660 (1944); cf. these Rev. 6, 225].

Haag, J. Sur le calcul de certaines intégrales au moyen de la fonction Γ. Bull. Sci. Math. (2) 65, 181–185 (1941). [MF 12837]

The integral

$$\int_0^1 x^m(1-x)^n(1-x^{\lambda_1})^{a_1}\cdots(1-x^{\lambda_q})^{a_q}dx,$$

where $m+1>0$, $n+a_1+\cdots+a_q+1>0$, $\lambda_i>0$, is evaluated as a sum of beta functions. *H. Pollard.*

Ionesco, D. V. Sur les formules généralisées de G. Darboux, et un théorème sur les quadratures mécaniques. Bull. Math. Soc. Roumaine Sci. 43, 7–22 (1941). [MF 12725]

This paper is a supplement to a previous note [Acad. Roum. Bull. Sect. Sci. 22, 165–167 (1939); these Rev. 1, 299]. In the first part the author gives references to an earlier paper by T. Popoviciu; in the second part it is shown that certain constants which appear in the solution of a linear system of equations are positive. The result has geometrical applications. *O. Szász* (Cincinnati, Ohio).

Pompeiu, D. Formes diverses du théorème des accroissements finis. Bull. École Polytech. Bucarest [Bul. Politehn. București] 13, 23–25 (1942). [MF 13562]

The author applies the law of the mean for derivatives to composite functions $F\{f(x)\}$ and indicates an application. *R. P. Boas, Jr.* (Providence, R. I.).

Bruwier, L. Sur la valeur moyenne des fonctions continues de plusieurs variables. Bull. Soc. Roy. Sci. Liége 9, 80–89 (1940). [MF 13038]

For the case of two functions of two variables, the author's result for p functions of n variables would read as follows. Let $f(x, y)$ and $g(x, y)$ be real-valued and continuous in a region R, and let $p+q+r+s=0$. Then there exists a point (x_0, y_0) in R such that

$$pg(x_0, y_0)\int\int f(x, y)+qf(x_0, y_0)\int\int g(x, y) +r\int\int f(x, y_0)g(x_0, y)+s\int\int f(x_0, y)g(x, y_0)=0,$$

integrations being with respect to x and y over R. For $g\equiv 1$, $p=1$, $q=-1$, $r=s=0$, this reduces to the familiar statement that $f(x, y)$ is somewhere equal to its average over R.

R. P. Boas, Jr. (Providence, R. I.).

Molland, Jacob. Généralisation d'un problème qui se rattache a l'étude d'une classe de réactions chimiques. Arch. Math. Naturvid. 46, no. 5, 139–154 (1943). [MF 12980]

The chemical problem leads to the equation

$$dt=\frac{dz}{\alpha_n{}^n(a-z)^n-\beta_n{}^nz^n}, \qquad z(0)=0,$$

for the concentration z of a product at time t, where n is a positive integer. The author shows that

$$\lim_{t\to\infty} z(t)=\frac{\alpha_n a}{\alpha_n+\beta_n},$$

unless $\alpha_n=\beta_n=0$. *R. P. Boas, Jr.* (Providence, R. I.).

Liebowitz, Benjamin. A theorem on functional determinants. Phys. Rev. (2) 68, 53–54 (1945). [MF 12877]

A determinant is set up, similar to a Jacobian, but with one modified row involving total derivatives for s, a preferred independent variable. The author proves that the logarithmic derivative of the determinant with respect to s equals the divergence of the vector whose n components are the elements of the modified row. *P. Franklin.*

Gillis, Paul. Sur la généralisation d'un théorème de L. Lichtenstein. Bull. Soc. Roy. Sci. Liége 11, 634–645 (1942). [MF 13126]

Let L, M, N be three functions of (x, y, z) which are defined in the regular region D with boundary fD and zero outside $D+fD$. The author generalizes three theorems of L. Lichtenstein [Grundlagen der Hydromechanik, Springer, Berlin, 1929, pp. 87, 96, 101]. The following is a typical theorem. Lichtenstein proved that, if L, M, N belong to class C^1 in $D+fD$ and on fD

$$(1)\qquad L\cos\alpha+M\cos\beta+N\cos\gamma=0,$$

while in $D+fD$

$$(2)\qquad L_x+M_y+N_z=0,$$

then there exist three functions P, Q, R such that R_y-Q_z, P_z-R_x, Q_x-P_y equal L, M, N in D and vanish outside $D+fD$. The author shows that, if L, M, N belong to C^0 in $D+fD$, (1) holds and (2) is replaced by

$$\int_{fc^3} Ldydz+Mdzdx+Ndxdy=0,$$

where fc^3 is the boundary of any regular region in space, then there exist three continuous functions P, Q, R such that

$$\int_{fc^2} Pdx+Qdy+Rdz=\int_{c^2} Ldydz+Mdzdx+Ndxdy,$$

$$\int_{fc^3} Pdydz+Qdzdx+Rdxdy=0,$$

where c^2 is any multiplicity of dimension 2 in space with boundary fc^2. *F. G. Dressel* (Durham, N. C.).

Theory of Functions of Complex Variables

***Knopp, Konrad. Theory of Functions. I. Elements of the General Theory of Analytic Functions.** Dover Publications, New York, 1945. vii+146 pp. $1.25.

This is a translation by Frederick Bagemihl of the fifth German edition [Göschen, Berlin-Leipzig, 1937], with a few improvements by the translator. In particular, the lemmas on the decomposition and triangulation of polygons, which were proved incorrectly in the German edition, are given new proofs which are correct. *L. H. Loomis.*

Bernstein, S. Constructive theory of functions as a development of Tchebychef's ideas. Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] 9, 145–158 (1945). (Russian. English summary) [MF 13325]

An expository lecture, presenting results and problems in that approach to the theory of functions which centers around (a) the best approximation, (b) interpolation by the least squares method, (c) the problem of moments.

A. Zygmund (Philadelphia, Pa.).

***Neville, Eric Harold. Jacobian Elliptic Functions.** Oxford University Press, 1944. xiii+331 pp. $7.50.

In a lecture in 1943 [J. London Math. Soc. 18, 177–191 (1943); these Rev. 5, 234], the author gave what amounts substantially to a preview of this volume. He remarked, "thirty-five years ago my contemporaries would have included in any such list [a list of mathematical topics which would have seemed as important to all future students as they did to the author's generation] the elements of the theory of the Jacobian elliptic functions; it never crossed our minds that the time would come when the ordinary mathematical undergraduate was to know less of these functions than we did." The author is of the opinion that the principal reason for this neglect of the subject can be attributed to "the unnatural way in which the theory is stifled by the multitude of special formulae to which its applications give rise." That this criticism is valid to a certain extent no one will deny. Yet it is doubtful that the lack of interest in the subject can be in a large measure ascribed to inadequate presentation. Mathematics, in common with other things of interest to the human race, is subject to fashions and styles; even the rigor in which it prides itself is relative and subject to change. A rapid examination of the mathematical abstracting and review journals indicates clearly that, over a period of fifty years, the theory of elliptic functions has become less and less fashionable. This is partly due to the fact that its theory was fairly thoroughly explored at an early stage and partly because other newer branches of analysis have forced it into the background, this in spite of its important contributions to geometry and to the theory of numbers. The volume under review is an

attempt to restore the Jacobian functions to the university curriculum by introducing them in a manner which is appealing by reason of its directness, simplicity and novelty.

As is well known, an elliptic function is a doubly-periodic meromorphic function of a complex variable z. The complex plane is divided into a lattice or mesh of period-parallelograms and the elliptic functions are defined on this lattice in the sense that, if the zeros and poles of such a function are prescribed within a period-cell, it is completely determined to within a multiplicative constant. Insofar as singularities are concerned, the simplest elliptic functions are those of "order" two and may, therefore, be classified into two categories: one includes all those functions which have a single irreducible double pole at which the residue is zero; the other contains all those functions which have two simple poles at which the residues have the same norms but have amplitudes differing by π. The first category is usually referred to as "Weierstrassian" and the function $\wp(z)$ is its prototype; the second category is similarly referred to as "Jacobian." Any elliptic function may be expressed in terms of either of these two types.

Weierstrass [Mathematische Werke, vol. 2, Berlin, 1895, pp. 245–255] defined the function $\wp(z)$ directly with reference to the given lattice. In contrast with this method of approach, the usual treatment of the Jacobian functions seems indirect since it is made to depend on the theory of the theta functions. This approach has many important practical and theoretical advantages but is undoubtedly rather sophisticated, and the author remarks, with considerable justification, that "a theory in which the definition of the fundamental function takes the form

$$\frac{\vartheta_3}{\vartheta_2}\,\frac{\vartheta_1(\mu/\vartheta_3^2)}{\vartheta_4(\mu/\vartheta_3^2)}$$

starts with a handicap of artificiality." In his book, therefore, the Jacobian functions are exhibited as functions constructed on a lattice and their theory is developed on the basis of the Weierstrassian function. This method of introducing these functions is not entirely an innovation, but used in conjunction with a new notation devised by the author it makes it possible to develop, in a surprisingly simple way, the systematic and structural relations among the functions. Thus, writing ω_f, ω_g, ω_h for the half-periods ω_1, ω_2, ω_3 of Weierstrass, the author defines three primitive elliptic functions by means of the relation

$$\operatorname{pj} z = \sqrt{\wp(z)-e_p}, \qquad \mathrm{p}=\mathrm{f}, \mathrm{g}, \mathrm{h};\ e_p=\wp(\omega_p),$$

with the conditions that the residue at the origin is unity. These three functions have a common pole at the origin and zeros at ω_f, ω_g, ω_h, respectively. To complete the notation, the symbol ω_j is used to represent the origin and the symbol pq z to denote a function with a simple zero at ω_p and a simple pole at ω_q.

The primitive triad just defined may be seen to be a section of a group of twelve functions, each of which has one of the four points ω_j, ω_f, ω_g, ω_h for a zero and one of the remaining three for a pole. Moreover, the introduction of a canonical or "Jacobian" lattice and of a normalizing factor leads to the identification of the twelve functions pq z with the twelve functions designated by Glaisher [Messenger of Math. 11, 81–95 (1881), in particular, p. 85] by cn u, sn u, etc. In fact, the functions fj z, gj z and hj z mentioned above are in effect the functions cs u, ns u and ds u, respectively.

The reader already initiated in the theory of elliptic functions may greet the introduction of still another notation with dismay and discouragement. However, this notation reflects both the structure of the functions and their relations to the Jacobian system, so that the development of the addition theorems and the discussion of the Landen and Jacobi transformations can be carried out with a considerable gain in clarity and insight.

The most original part of the book consists of those chapters which deal with the difficult "inversion problem." This problem has been treated with great care and thoroughness. The nature and importance of the problem are clearly outlined and the solution given is developed along lines which, to the reviewer, seem to be entirely new, involving an application of the theory of point sets and, in particular, of the Heine-Borel theorem.

For the sake of completeness, the author has included a chapter on the theta functions. In harmony with the general point of view adopted in the discussion of the elliptic functions, he introduces the theta functions as integral functions with a specified lattice of zeros. The reader must be warned that here, again, a departure from the more standard notation is encountered. Thus the functions $\vartheta_s(u)$, $\vartheta_c(u)$, $\vartheta_n(u)$, $\vartheta_d(u)$ are constant multiples of Jordan's $\theta(u)$, $\theta_1(u)$, $\theta_2(u)$, $\theta_3(u)$, respectively.

A collection of annotated exercises is given at the end of the book. Some of these lead to other proofs of theorems in the text and to additional results; others involve excursions beyond the range of the text. Unfortunately, the book is not provided with an index. *M. A. Basoco.*

Bohr, Harald. On power series with gaps. A pseudo-continuity property. Mat. Tidsskr. B. 1942, 1–11 (1942). (Danish) [MF 13585]

If m_n is an increasing sequence of positive integers such that $\sum m_n/m_{n+1}$ converges, the (unbounded) function $f(z)=\sum_{n=1}^{\infty} z^{m_n}$ has the property that for every positive ϵ there are arbitrarily small numbers τ for which

$$|f(ze^{i\tau})-f(z)|\leqq\epsilon, \qquad |z|<1.$$

R. P. Boas, Jr. (Providence, R. I.).

Mandelbrojt, S. Sur un théorème de M. Whittaker. Bull. Soc. Math. France 69, Communications et Conférences 1–2 (1941). [MF 13244]

The author points out that a theorem of J. M. Whittaker [J. London Math. Soc. 13, 295–301 (1938)] can be generalized, using Whittaker's method, as follows. Let $f(z)=\sum a_{\lambda_n} z^{\lambda_n}$ have radius of convergence 1 and have, on $|z|=1$, only k singular points, all isolated and noncritical, and all on an arc of length α. Let p be an integer less than or equal to $2\pi/\alpha$, and let $\mu_n^{(p)}$ denote the λ_n which are congruent to p (mod $[2\pi/\alpha]$). Then $\limsup \mu_n^{(p)}/n\leqq 2\pi k/\alpha$. This reduces to Whittaker's theorem if $\alpha=2\pi$ and $p=0$.

R. P. Boas, Jr. (Providence, R. I.).

Alenitzyn, G. On the coefficients of "schlicht" functions. Rec. Math. [Mat. Sbornik] N.S. 15(57), 131–138 (1944). (Russian. English summary) [MF 12284]

Suppose that the function $f(z)=z+\sum_2^{\infty} c_n z^n$ is regular and schlicht in $|z|<1$. Let m be an integer greater than 1 and write

$$f_m(z)=z+c_{m+1}z^{m+1}+c_{2m+1}z^{2m+1}+\cdots,$$

$f_{(m)}(z)=f(z)-f_m(z)$. The author, generalizing the method used in the case $m=2$ by Basilevitch [same Rec. N.S. 6(48),

337–344 (1939); these Rev. 1, 308], proves that, if

$$|f'_{(m)}(z)|=O\{(1-|z|)^{-(m+1)/m+\epsilon}\},\qquad \epsilon>0,\ |z|<1,$$

then $|f(z)|=O\{(1-|z|)^{-2/m}\}$. In particular, if we suppose that the coefficients of $f_{(m)}(z)$ are $O(n^{-(m-1)/m-\epsilon})$, the hypothesis concerning $|f'_{(m)}(z)|$ is satisfied and, in the cases $m=2$ and 3, the conclusion implies (as is well known) that the coefficients of $f_m(z)$ are $O(n^{2/m-1})$. *D. C. Spencer.*

Gelfer, S. Sur les bornes de l'étoilement et de la convexité des fonctions p-valentes. Rec. Math. [Mat. Sbornik] N.S. 16(58), 81–86 (1945). (Russian. French summary) [MF 13015]

The author considers the following three classes of functions: (1) the class Σ_p of functions

$$w=F(z)=\sum_{n=-p}^{\infty}\alpha_n z^n,\qquad \alpha_{-p}=1,$$

which for $|z|<1$ are p-valent, nonvanishing and regular (except for a pole of order p at $z=0$); (2) the subclass Σ_p' of Σ_p for which $\alpha_{-p+1}=0$; (3) the class S_p of functions

$$w=f(z)=\sum_{n=p}^{\infty}a_n z^n,\qquad a_p=1,$$

which are p-valent and regular for $|z|<1$. If F belongs to Σ_p, then $1/F$ belongs to S_p and conversely. A curve in the w-plane is called starlike if the argument of w is nondecreasing or nonincreasing as the curve is traversed in some sense; convex if the tangent to the curve always turns in the same sense as the curve is described. The radius of starlikeness (or convexity) for the class considered is defined to be the least upper bound of numbers $r>0$ for which $|z|=r$ is mapped by each function of the class into a starlike (or convex) curve. Let R_s and R_k be the radii of starlikeness and convexity, respectively, for the class Σ_p, R_s' the radius of starlikeness for the class Σ_p', ρ_s the radius of starlikeness for the class S_p. The author proves (a) $3^{-\frac{1}{2}}\leqq\rho_s=R_s\leqq\tanh \pi/4$, (b) $R_s'\geqq 2^{-\frac{1}{2}}$, (c) $R_k>0.55$.
D. C. Spencer (Stanford University, Calif.).

Ríos, Sixto. An elementary proof of the fundamental theorem of normal families. Fac. Ci. Madrid. Publ. Sem. Mat. no. 1, 6 pp. (1941). (Spanish) [MF 12803]

The fundamental theorem concerning normal families of analytic functions states that, if the members of a family of functions analytic in a domain D all omit two finite values a and b, then the family is normal. The proofs usually given involve consideration of the modular function or other analogous non-elementary functions. A demonstration has been given by Valiron [C. R. Acad. Sci. Paris 183, 728–730 (1926)], involving Bloch's theorem, the Landau-Schottky theorem, a theorem of Borel, and a transformation used by Bloch in establishing the theorem of Picard. A proof is now given which is simpler than that of Valiron in that, of the above tools, only Bloch's theorem and Bloch's transformation are used. *E. F. Beckenbach* (Los Angeles, Calif.).

Lamoen, J. Sur la correspondance point par point de deux arcs circulaires qui sous-tendent des angles différents. Bull. Soc. Roy. Sci. Liége 12, 492–496 (1943). [MF 13152]

Fuchs, B. Sur la fonction minimale d'un domaine. I. Rec. Math. [Mat. Sbornik] N.S. 16(58), 21–38 (1945). (Russian. French summary) [MF 12620]

In order to generalize the methods of conformal mapping to mappings by n functions of n complex variables, Bergman introduced, in the case of one variable, a metric $ds_B^2(z)=\{\partial^2\log\kappa_B/\partial z\partial\bar z\}|dz|^2$, which is invariant with respect to conformal transformations. Here $\kappa_B=\sum_{\nu=1}^{\infty}|\varphi_\nu(z)|^2$, where $\{\varphi_\nu\}$ is a complete set of functions which are orthonormal in B [see J. Reine Angew. Math. 169, 1–42 (1933)]. Let $S(B,t)$ be the totality of domains which one obtains by functions which are normed to have the value 1 at the point t. The transformation $w=\kappa_B(z,\bar t)/\kappa_B(t,\bar t)$ defines the mapping of B into the "representative domain" of $S(B,t)$ [Math. Ann. 102, 430–446 (1930)]. Continuing an earlier paper [same Rec. N.S. 2(44), 567–594 (1937)], the author investigates certain quantities which depend on two points of the domain, are invariant with respect to conformal transformation, satisfy the triangle inequality, and vanish when the points coincide. (They can thus be used as the distance between two points.) Let $r=1-u^{\frac{1}{2}}$, where

$$u=\kappa_B(z,\bar t)\kappa_B(t,\bar z)/\kappa_B(z,\bar z)\kappa_B(t,\bar t)$$

and κ_B is the function described above. Let $\varphi(z,t)=(2r)^{\frac{1}{2}}\varphi(Ir)$, where $\varphi(u)$ is a differentiable function, $\varphi(0)=\varphi'(0)=1$, and $I\equiv I(z,t)$ is an invariant of the metric (for example, a constant). The author derives the following results. (I) Let Γ be a curve defined by functions having third derivatives, R the Gaussian curvature of the invariant metric, C_Γ the geodesic curvature of Γ in the same metric, and let $Q=C_\Gamma^2-R-12I+3\neq0$ at some point $z=z_0$ of Γ. Then there exists a neighborhood of z_0 such that for z_1, z_2 and z_3 belonging to it we have $(\operatorname{sgn} Q)\{\varphi(z_1,z_2)+\varphi(z_2,z_3)-\varphi(z_1,z_3)\}>0$. (II) Under the same conditions

$$r_{12}^{\frac{1}{2}}\varphi(I_{12}r_{12})+r_{23}^{\frac{1}{2}}\varphi(I_{23}r_{23})\geqq r_{13}^{\frac{1}{2}}\varphi(I_{13}r_{13}),$$

where $r_{mn}=r(z_m,z_n)$, $I_{mn}=I(z_m,z_n)$. (III) Under the same conditions

$$(\operatorname{sgn} Q)\{\kappa_B(z,\bar t)-(\kappa_B(z,\bar z)\kappa_B(t,\bar t))^{\frac{1}{2}}(1-I^{-1}f(((\tfrac{1}{2}I)^{\frac{1}{2}}\sigma_\Gamma))\}\geqq0.$$

Here f is any function with sufficiently many derivatives, $z\epsilon\Gamma$, $t\epsilon\Gamma$ and σ_Γ is the (invariant) distance between z and t. (IV) Under the same conditions

$$\kappa_B(z,\bar t)\geqq\{\kappa_B(z,\bar z)\kappa_B(t,\bar t)\}^{\frac{1}{2}}\{1-\tfrac{1}{2}\sigma_\Gamma^2\}^{\frac{1}{2}}.$$

Here t is a point sufficiently near z. The author derives analogous inequalities for the case where $Q=0$ at z. He also shows that through every point of B there pass two curves such that for any pair of points (z,t) on each of the curves $\sigma_\Gamma(z,t)=\varphi(z,t)$. These results lead to various conclusions in the theory of conformal mapping of multiply connected domains, in particular, concerning the distortion of Euclidean measures. *S. Bergman* (Providence, R. I.).

Ferrand, Jacqueline. Sur l'inégalité d'Ahlfors et son application au problème de la dérivée angulaire. Bull. Soc. Math. France 72, 178–192 (1944). [MF 13227]

The author reviews and ameliorates results of her thesis [Ann. Sci. École Norm. Sup. (3) 59, 43–106 (1942); these Rev. 6, 207] concerning the existence of angular derivatives for one-to-one directly conformal maps.
M. H. Heins (Providence, R. I.).

Dufresnoy, Jacques. Sur la correspondance des frontières dans la représentation conforme. C. R. Acad. Sci. Paris 220, 189–190 (1945). [MF 13494]

This note contains a summary of the results of the paper reviewed below. *M. H. Heins* (Providence, R. I.).

Dufresnoy, Jacques. Sur les fonctions méromorphes et univalentes dans le cercle unité. Bull. Sci. Math. (2) **69**, 21–36 (1945). [MF 13626]

This paper establishes two theorems concerning functions $w=f(z)$ which map $|z|<1$ directly conformally and one-to-one onto a simply-connected region D of the w-plane. The first theorem deals with the set $E_1(P)$ of points of $|z|=1$ associated by the mapping $w=f(z)$ with those prime ends of the boundary of D which contain a given point P of the w-plane. It is shown that the set $E_1(P)$ (assumed nonvacuous) is closed and of capacity zero. The proof is reduced by a series of preliminary mappings to the study of a Green's function defined in a region consisting of the extended plane deleted in a set of segments on the real axis. The theorem is then established by examining the properties of this Green's function. The second theorem, whose proof is in the same spirit as the first, concerns the set E_2 of points of $|z|=1$ associated with prime ends which either do not possess accessible points or else have points accessible only by nonrectifiable paths. The theorem states that the closure of E_2 has capacity zero. *M. H. Heins.*

Dufresnoy, Jacques. Sur les domaines couverts par les valeurs d'une fonction méromorphe ou algébroïde. Ann. Sci. École Norm. Sup. (3) **58**, 179–259 (1941). [MF 12895]

Let F_0 be a closed surface of the topological type of a sphere, and let notions of length and area be defined on F_0, satisfying certain simple conditions. If F is any closed covering surface of F_0 and D is a simply connected domain on F_0, we define an island over D as a connected domain on F over D which has no boundary relative to D, and the simple multiplicity of the island as the negative of its characteristic. Let $D_1, \cdots, D_q$ $(q\geqq 3)$ be mutually disjoint simply connected domains on F_0, and let $p(D_i)$ be the sum of the simple multiplicities of the islands of F over D_i; Ahlfors [Acta Math. **65**, 157–194 (1935)] proved that, if F is simply connected, then $\sum_{i=1}^{q}p(D_i)\geqq(q-2)S-hL$, where L is the length of the boundary of F, S is the area of F divided by that of F_0, and h depends only on $F_0, D_1, \cdots, D_q$. In the present paper a new proof is given for the case where F_0 is the unit sphere and the ordinary notions of length and area are used. The author's method enables him to estimate h explicitly. He also obtains an extension to multiply connected covering surfaces.

If the domains D_i reduce to points, $h=3/(2\delta_0)$, where δ_0 is the smallest of the spherical distances between these points. Here $p(D_i)$ is replaced by the number of points of F over a_i, each counted only once. If the domains are circular or convex and δ_0 is the smallest spherical distance between any two of them, then $h<3\pi/(2\delta_0^2)$. As a consequence, if every island over D_i has at least μ_i sheets, then $L\geqq h'\{\sum(1-1/\mu_i)-2\}S$, where h' depends only on $D_1, \cdots, D_q$. Here again the author gives explicit estimates for h'. Let $w=f(z)$ be meromorphic in $|z|<R$ and let $D_1, \cdots, D_q$ be as above on the w-sphere; let μ_i be defined as above for the Riemann surface of the inverse of $f(z)$. If $\sum(1-1/\mu_i)>2$, then $R<\infty$ and $R|f'(0)|/(1+|f(0)|^2)<K$, where K depends only on the domains D_i and the numbers μ_i and can be estimated explicitly. This leads to numerical bounds (of course, not the best possible) in the classical theorems of Landau, Schottky, Koebe and Bloch and also to explicit bounds in generalizations. In Schottky's theorem, for example, the bound obtained for $\log|f(z)|$ in $|z|\leqq r<1$ for a function $f(z)\neq 0$, $\neq 1$ and analytic in $|z|<1$ is $O((1-r)^{-1}\log(1-r)^{-1})$; the best bound [Robinson, Bull. Amer. Math. Soc. **45**, 907–910 (1939); these Rev. **1**, 112] is $O((1-r)^{-1})$. New criteria for normal families are obtained.

The case where there are at most a fixed number d of exceptional islands with less than μ_i sheets is also studied and similar results are obtained. Extensions to algebroid functions related to the work of Valiron and Remoundos are made. Let $w=f(z)$ be an algebroid function in $|z|<R$ with ν branches whose determinations are distinct at the origin and such that one branch has a spherical oscillation greater than δ_1 in $|z|\leqq r_1<R$. Suppose, furthermore, that there are q $(\geqq 2\nu+1)$ simply connected mutually disjoint domains D_i on the w-sphere such that every island over D_i of the Riemann surface of the inverse function has at least μ_i sheets and that $\sum(1-1/\mu_i)>2\nu$. Then $R<\infty$ and $\log(R\delta_1^{\nu}/r_1)<F(\prod\delta_{ij}(0), h', \sum(1-1/\mu_i), \nu)$, where $\prod\delta_{ij}(0)$ is the product of the spherical distances of the branches of $f(z)$ at the origin, and h' is a constant depending only on the domains D_i. In the case $\nu=2$, the author shows that the term $\prod\delta_{ij}(0)$ is essential in general but can be eliminated under certain conditions. The principal novelty in the method is the systematic use of isoperimetric inequalities. *P. C. Rosenbloom* (Providence, R. I.).

Dufresnoy, Jacques. Sur les valeurs ramifiées des fonctions méromorphes. Bull. Soc. Math. France **72**, 76–92 (1944). [MF 13220]

Using the notations of the preceding review, let condition C be that every island over D_i $(1\leqq i\leqq q, q\geqq 3)$ with at most d_i exceptions has at least μ_i sheets. Suppose it is impossible for the Riemann surface of the inverse of a function meromorphic in the whole plane to satisfy C. Then the class of functions $f(z)$ meromorphic in $|z|<R$ $(R<\infty)$ and such that the corresponding Riemann surface of the inverse of $f(z)$ satisfies condition C is a normal family. This contains many classical results as special cases. The analogous result holds with condition C replaced by $\bar{C}$: the number of islands over D_i with at most k sheets is at most $d_i^{(k)}$ $(1\leqq k<\mu_i)$. These results are based on a new theorem on covering surfaces. The methods used would enable one to obtain explicit bounds in theorems of the Schottky type under the more general conditions. *P. C. Rosenbloom.*

Dufresnoy, Jacques. Sur l'aire sphérique décrite par les valeurs d'une fonction méromorphe. Bull. Sci. Math. (2) **65**, 214–219 (1941). [MF 13260]

In this note the following lemma, which is closely related to a lemma proved previously by the author [C. R. Acad. Sci. Paris **212**, 662–665; these Rev. **3**, 81], is established. Let $w=f(z)$ be meromorphic for $|z|<r_0$ and let $4\pi\sigma(r)$ denote the spherical area of the closed simple region on the Riemann sphere each of whose points is covered at least once by the image of $|z|\leqq r(<r_0)$ with respect to $w=f(z)$. If $\lim_{r\to r_0}4\pi\sigma(r)=4\pi\sigma_0<4\pi$, then

$$r^{-2}\sigma(r)/(1-\sigma(r))\leqq r_0^{-2}\sigma_0/(1-\sigma_0). \tag{1}$$

This relation yields

$$|f'(0)|^2/\{1+|f(0)|^2\}^2\leqq r_0^{-2}\sigma_0/(1-\sigma_0) \tag{2}$$

(properly interpreted in the case of a pole at $z=0$). The proof is based upon isoperimetric inequalities relating spherical length and area as well as on Schwarz's inequality. *M. H. Heins* (Providence, R. I.).

Rauch, A. Deux remarques sur les fonctions entières d'ordre fini ρ. Bull. Soc. Math. France 72, 93–96 (1944). [MF 13221]

Rauch, A. Sur les directions de Borel des fonctions entières de la classe de divergence de l'ordre ρ. Bull. Sci. Math. (2) 65, 219–224 (1941). [MF 13261]

These papers are concerned with the following and related results. Let $f(z)$ be meromorphic of order ρ and

$$I(\theta)=\lim_{k\to\rho+0}\int_1^\infty \log^+ f(re^{i\theta})r^{-k-1}dr\Big/\int_1^\infty m(r,f)r^{-k-1}dr.$$

Then $\sum_j\{I'(\theta_j+0)-I'(\theta_j-0)\}=2\pi\rho^2$, where the θ_j are the Borel directions of $f(z)$. *O. Helmer* (New York, N. Y.).

Bellman, Richard. A note on mean values. Univ. Nac. Tucumán. Revista A. 4, 255–258 (1944). [MF 13028]

The author attempts to prove that, if $|f(z)|\leqq Ae^{B|z|}$, then

$$\lim_{T\to\infty} T^{-1}\int_0^T |f'(x)|^p dx\leqq R^p \lim_{T\to\infty} T^{-1}\int_0^\infty |f(x)|^p dx.$$

The proof is incorrect. *R. P. Boas, Jr.*

Myrberg, P. J. Über analytische Funktionen auf transzendenten zweiblättrigen Riemannschen Flächen mit reellen Verzweigungspunkten. Acta Math. 76, 185–224 (1945). [MF 13200]

Let F be a two-sheeted Riemann surface which is ramified at the real points e_ν $(\nu=0, 1, \cdots)$ such that $\lim_\nu e_\nu=\infty$. The author considers F as the limit of the hyperelliptic surfaces F_p of genus p which are ramified at $e_0, \cdots, e_{2p-1}$, and considers first the integrals on the surfaces F_p. Let A_ν, B_ν $(1\leqq\nu\leqq p)$ be canonical retrosections of F_p, where the B_ν are conjugate to the A_ν lying in the upper sheet of F_p over the segments $[e_{2\nu-1}, e_{2\nu}]$, $[e_{2p-1}, \infty, e_0]$. The cuts B_ν are selected so that $\lim_{p\to\infty} B_\nu$ is the ideal boundary of F. Then the infinite systems $A_1, A_2, \cdots, B_1, B_2, \cdots$ furnish a canonical set of retrosections for F. Let F' be the associated simply connected surface. The author considers real integrals in F where the integrand assumes real values for real x and y. A normal integral of the first kind on F is defined to be a real integral $\varphi=u+iv$ such that $|v|\leqq M<\infty$ on F'. In order to construct special normal integrals of the first kind recourse is taken to the integrals $\varphi_n^{(p)}=u_n^{(p)}+iv_n^{(p)}$ of the first kind on F_p. The $\varphi_n^{(p)}$ are chosen to vanish at e_0 and so that $\int_{A_n}d\varphi_n^{(p)}=2\pi i$, $\int_{A_\nu}d\varphi_n^{(p)}=0$ $(\nu\neq n)$. Then the periods $\int_{B_\nu}d\varphi_n^{(p)}=\tau_{n\nu}^{(p)}$ are real. The limit $\lim_{p\to\infty} v_n^{(p)}=v_n$ is defined as a harmonic function on F. If u_n is the conjugate harmonic function with $u_n(e_0)=0$, then $\varphi_n=u_n+iv_n=\lim_{p\to\infty}\varphi_n^{(p)}$ is finite on F and (i) $\int_{A_n}d\varphi_n=2\pi i$, $\int_{A_\nu}d\varphi_n=0$ $(\nu\neq n)$, $\int_{B_\lambda}d\varphi_n=\tau_{n\lambda}=\lim_{p\to\infty}\tau_{n\lambda}^{(p)}$, (ii) $|v_n|\leqq\pi$ on F'. These integrals are shown, using the hyperelliptic integrals, to form a basis for the normal integrals of the first kind. Conversely, each sum $\sum_{n=1}^\infty c_n\varphi_n$ with bounded sums $\sum_{n=1}^k c_n$ represents a normal integral of the first kind.

A real integral χ on F is called a general integral of the third kind if (i) χ is regular on F except for a set of regularly distributed points, on which χ has logarithmic singularities with the residues $+1$, -1 and $\chi(e_0)=0$, (ii) $|\Im\chi|\leqq M<\infty$ on F'. Here a set of real points (a_ν, b_ν) with $\lim a_\nu=\lim b_\nu=\infty$ is called regularly distributed if (a_ν, b_ν) is in some interval $C_\mu=[e_{2\mu-2}, e_{2\mu-1}]$ and each interval C_μ contains a bounded number $N_\mu\leqq N<\infty$ of pairs (a_ν, b_ν). Using singular normal integrals of the third kind (limits of hyperelliptic integrals of the third kind with the residue $+1$ at finite distance and the residue -1 at ∞), the following generalization of a part of Abel's theorem holds. If (a_ν, b_ν), $\nu=1, 2, \cdots$, are the conjugate real poles and zeros of f, then

$$\sum[\varphi_\nu(b_i)-\varphi_\nu(a_i)]=\sum m_i\tau_{i\nu}+2\pi ik_\nu, \quad \nu=1, 2, \cdots,$$

with integers m_i, k_ν.

After these preparations the author shows the existence of an integral of the first kind $\int(h(x)/y)dx=I$, $h(x)$ an entire function, such that the periods

$$\int_{A_\nu}dI=2\pi i\rho_\nu, \quad \int_{B_\nu-B_{\nu-1}}dI=2\pi i\bar{\rho}_\nu$$

are given numbers. Finally, it is shown how entire functions on F with given real zeros can be constructed. The method involves a generalization of the prime functions in the classical theory of algebraic functions.

O. F. G. Schilling (Chicago, Ill.).

Morse, Marston, and Heins, Maurice. Topological methods in the theory of functions of a single complex variable. I. Deformation types of locally simple plane curves. Proc. Nat. Acad. Sci. U. S. A. 31, 299–301 (1945). [MF 13435]

This is a summary of the first of a series of papers on interior transformations and related subjects. It defines the necessary topological concepts associated with closed oriented curves. A closed oriented curve g is called locally simple if there is a positive constant e such that any point P of g is an interior point of a simple subarc of g whose end-points P_1, P_2 satisfy $\overline{P_1P}>e$, $\overline{P_2P}>e$. The angular order of g is an integer p defined as follows. If g is given as the locally one-to-one continuous image of a unit circumference $u+iv=f(\theta)$, then for ϵ positive and less than a sufficiently small constant e_1, the change in angle of $f(\theta+\epsilon)-f(\theta)$ as θ increases from 0 to 2π is $2p\pi$, independent of ϵ. If g is deformed through a continuous family of uniformly locally simple curves, then the angular order p is the sole invariant. If g does not pass through a point O, and if only deformations of g are admitted which never pass through O, there is the additional invariant $q(O)$, which is the order of g relative to O. Normal forms of g are given, consisting of circles repeated a specified number of times.

M. Shiffman (New York, N. Y.).

Morse, Marston, and Heins, Maurice. Topological methods in the theory of functions of a complex variable. II. Boundary values and integral characteristics of interior transformations and pseudo-harmonic functions. Proc. Nat. Acad. Sci. U. S. A. 31, 302–306 (1945). [MF 13436]

This second summary [see the preceding review] studies the topology of meromorphic functions or, more generally, interior transformations. Let G be a region in the z-plane bounded by ν Jordan curves (which may be taken as circles), and let $w=f(z)$ be a meromorphic function, defined over G, which is continuous on the boundaries of G and maps these boundaries into locally simple curves in the w-plane not passing through the origin. The following inequality for the number μ of ramification points is stated:

$$(*)\qquad n(0)+n(\infty)-\mu\geqq 2-\nu+\sum q-\sum p,$$

where $n(0)$ is the number of zeros of $f(z)$ in the interior of G, $n(\infty)$ is the number of poles, μ is the number of branch points of the inverse of $f(z)$ (all counted with their multiplicities) and q and p are the orders and angular orders of

the images of the boundary curves in the w-plane. Equality holds in (*) if a neighborhood of each boundary point of G is mapped in a one-to-one way. Similar results apply if $w=f(z)$ is merely restricted to be an interior transformation rather than a meromorphic function. [The authors wish to state that through an oversight it was not stated in the paper that the interior transformations are sense-preserving.] Special cases of these results had been obtained previously by Stoïlow [C. R. Acad. Sci. Paris 190, 251–253 (1930)] and by Ålander [ibid. 184, 1411–1413 (1927)].

The authors consider other functions related to interior transformations. A function $U(x, y)$ defined over G is called pseudo-harmonic if, in the neighborhood of every interior point of G, it is the real part of an interior transformation. Relations are obtained between the number of logarithmic poles of U, the number of saddle points of U and the behavior of U on the boundary of G. *M. Shiffman.*

Rey Pastor, J. Complex functions in topological space. Univ. Nac. Tucumán. Revista A. 4, 159–216 (1944). (Spanish) [MF 13024]

This paper contains the foundations of a theory of continuous functions from a Hausdorff space E_z to the complex w-plane; in most of the paper, E_z is a closed two-dimensional orientable manifold. Two points of E_z are said to be at the same level if they are inverse images of the same point w_0; a limit point of points at the same level is called a point of striction. The index of a region, bounded by a curve C, with respect to the point a, is $(2\pi)^{-1}\Delta_C \arg (w-a)$; the index of a point is the index of regions containing it but no pole or point of striction. Univalent and multivalent functions are defined in the natural way. Singular points are classified as points of striction and points of retrocession, that is, points z_0 such that the image of a neighborhood of z_0 contains the image w_0 of z_0 on its boundary. A point is regular if to every neighborhood of z_0 corresponds a neighborhood of w_0, and if the same holds for all points in a neighborhood of z_0.

These and other concepts are studied in their interrelations and illustrated by examples; portions of many classical theorems remain true in the more general framework. A function is topological on an open set Z if it is continuous and transforms Z into an open set. For topological functions, the maximum principle holds in Lindelöf's form: if to every positive ϵ and every boundary point there is a neighborhood in which $|f(z)|\leq M+\epsilon$, then $|f(z)|\leq M$ in Z. The topological functions on Z which are also topological on every subregion of Z coincide with the interior transformations of Stoïlow. A less restricted class is that of monotone functions, for which each point is regular and the index of each point has the same sign. Some properties of monotone functions are studied. *R. P. Boas, Jr.*

Hua, Loo-Keng. Geometries of matrices. I. Generalizations of von Staudt's theorem. Trans. Amer. Math. Soc. 57, 441–481 (1945). [MF 12564]

The analogy between the module of complex numbers and that of symmetric matrices, as it was first worked out by Siegel [Amer. J. Math. 65, 1–86 (1943); these Rev. 4, 242] and completed by the present author [Amer. J. Math. 66, 470–488, 531–563 (1944); these Rev. 6, 124], is carried a step further in a geometrical direction. The main theorem is the following generalization of von Staudt's theorem: A transformation satisfying the following conditions: (1) one-to-one and continuous, (2) carrying symmetric matrices into symmetric matrices, (3) keeping arithmetic distance invariant, (4) keeping the harmonic relation invariant, is either a symplectic or an anti-symplectic transformation. The definitions and notations are taken over from the author's previous papers. Added to those definitions is that of the "arithmetic distance" between two symmetric matrices W, Z, meaning the rank of $W-Z$. Moreover, a quadruple of symmetric matrices Z_1, Z_2, Z_3, Z_4 is said to form a harmonic range if

$$(Z_1-Z_3)(Z_1-Z_4)^{-1}(Z_2-Z_4)(Z_2-Z_3)^{-1}=-I.$$

The main theorem is implied by the following theorem. Let Φ be the complex field and let $\mathfrak{M}$ be a module formed by symmetric matrices over Φ. Let Γ be an additive continuous automorphism of $\mathfrak{M}$ leaving the rank invariant, so that Γ satisfies. (i) $\Gamma(X)\epsilon\mathfrak{M}$, if $X\epsilon\mathfrak{M}$; (ii) $\Gamma(X+Y)=\Gamma(X)+\Gamma(Y)$, if $X\epsilon\mathfrak{M}$ and $Y\epsilon\mathfrak{M}$; (iii) $\Gamma(iX)=i\Gamma(X)$; (iv) $\Gamma(X)$ has the same rank as X. Then $\Gamma(X)$ is an inner automorphism, that is, $\Gamma(X)=TXT'$ for certain T.

Similar investigations are then devoted to the geometry of skew-symmetric matrices (of even order) and to Hermitian matrices. Although the results are analogous, each of these two cases presents a particular difficulty. For skew-symmetric matrices of rank $n=2m=4$ there has to be considered the special automorphism

$$\Gamma\begin{pmatrix} P & Q \\ -Q' & R\end{pmatrix}=\begin{pmatrix} P & Q' \\ -Q & R\end{pmatrix},$$

which is not an inner one. In the case of Hermitian matrices the customary theory needs to be supplemented by a classification according to "signatures," a task which the author has carried out in one of the papers referred to above. [The letter $\mathfrak{T}$, which stands in the paper for a variable matrix, is erroneously used on pp. 442, 459 for the matrix

$$\begin{pmatrix} I & 0 \\ 0 & I\end{pmatrix},$$

which should, according to the author's previous papers, bear the name $\mathfrak{J}$.] *H. Rademacher* (Philadelphia, Pa.).

Hua, Loo-Keng. Geometries of matrices. I$_1$. Arithmetical construction. Trans. Amer. Math. Soc. 57, 482–490 (1945). [MF 12565]

This paper deals with the redundancy of some conditions of the main theorem of paper I [cf. the preceding review]. In the geometry of 2-rowed symmetric matrices it turns out that the invariance of the harmonic relation is a consequence of the invariance of the arithmetical distance. *H. Rademacher* (Philadelphia, Pa.).

Ostrowski, Alexandre. Addition à notre mémoire: "Recherches sur la méthode de Graeffe et les zéros des polynômes et des séries de Laurent." Acta Math. 75, 183–186 (1943). [MF 13209]

San Juan, R. À propos du mémoire: "Recherches sur la méthode de Graeffe . . . etc." par Alexandre Ostrowski, à Bâle. Acta Math. 75, 187–190 (1943). [MF 13210]

Discussion of connections among results of Ostrowski [same Acta 72, 99–155, 157–257 (1940); these Rev. 1, 323; 2, 342], Rey Pastor [Lecciones de Álgebra, 2d ed., Madrid, 1935, pp. 89–105], and San Juan [Revista Mat. Hisp.-Amer. (3) 1, 1–14 (1939); these Rev. 2, 61]. Errata to the first paper cited.

Kasner, Edward, and DeCicco, John. The geometry of polygenic functions. Univ. Nac. Tucumán. Revista A. 4, 7–45 (1944). [MF 13017]

The authors study fundamental geometric properties of the first and second derivatives of general polygenic functions, that is, of functions $w=\varphi(x, y)+i\psi(x, y)$ having continuous partial derivatives of the required orders with respect to x and y in a domain in the z-plane, where $z=x+iy$. Mean and phase derivatives are defined and discussed, as well as the first derivative dw/dz (which depends not only on the point z but also on the inclination θ at z of the curvilinear path C with respect to which the derivative is taken) and the second derivative d^2w/dz^2 (which depends on z and θ and also on the curvature κ of C). Several theorems are established concerning the first and second derivatives and the geometry of the related circles, limaçons, and cardioids. Numerous bibliographical references help make the paper a good introduction to the theory of polygenic functions. *E. F. Beckenbach* (Los Angeles, Calif.).

De Cicco, John. Survey of polygenic functions. Scripta Math. 11, 51–56 (1945). [MF 12777]

The author summarizes the results presented in the paper reviewed above. *E. F. Beckenbach* (Los Angeles, Calif.).

Fourier Series and Generalizations, Integral Transforms

Menchoff, D. Sur la convergence uniforme des séries de Fourier. Rec. Math. [Mat. Sbornik] N.S. 11(53), 67–96 (1942). (French. Russian summary) [MF 12824]

A detailed proof of a result announced, with a sketch of the proof, in C. R. (Doklady) Acad. Sci. URSS (N.S.) 32, 245–246 (1941); these Rev. 3, 106. *A. Zygmund.*

Kuttner, B. On the Riesz means of a Fourier series. II. J. London Math. Soc. 19, 77–84 (1944). [MF 12553]

The purpose of the present paper is to complete the results proved in part I [same J. 18, 148–154 (1943); these Rev. 5, 237] on the Riesz means of the Fourier series of a positive function. The complete results depend on the following theorem. If $0<\lambda<2$, there is a (finite) function $k(\lambda)$ such that the means (R, n^λ, κ) of the series $\frac{1}{2}+\sum_1^\infty \cos nx$ are all everywhere positive if $\kappa\geqq k(\lambda)$, but not if $\kappa<k(\lambda)$. The function $k(\lambda)$ is (strictly) increasing, continuous in $0<\lambda<2$, and tends to infinity as $\lambda\rightarrow 2$. Further, $k(1)=1$ and, if we denote the limit of $k(\lambda)$ as $\lambda\rightarrow 0$ by $k(0)$, then $0<k(0)<1$. If $\lambda=2$, the means (R, n^λ, κ) of the above series are not, for any κ, all everywhere positive, but the Abel means (A, n^λ) are. If $\lambda>2$, neither the Riesz means (R, n^λ, κ) nor even the Abel means (A, n^λ) are all everywhere positive. *R. Salem* (Cambridge, Mass.).

Kharchiladzé, Philippe. Sur la méthode de sommation de S. N. Bernstein. Rec. Math. [Mat. Sbornik] N.S. 11(53), 121–148 (1942). (Russian. French summary) [MF 12826]

The author extends certain theorems of S. Bernstein [C. R. Acad. Sci. Paris 191, 976–979 (1930)], but is apparently unaware of the papers of W. Rogosinski [Math. Ann. 95, 110–134 (1925); Math. Z. 25, 132–149 (1926)], which contain a number of the results of the present paper. Let $f(x)\sim\frac{1}{2}a_0+\sum_1^\infty(a_\nu \cos \nu x+b_\nu \sin \nu x)$ be the Fourier series of a function $f(x)$ and let $S_n(x)$ be the nth partial sum of the series. The author says that the series is summable at the point x by the method of Bernstein to sum $f(x)$, if as $n\rightarrow\infty$

$$(*)\quad \tfrac{1}{2}\{S_n(x+\pi/(2n+1))+S_n(x-\pi/(2n+1))\}$$
$$=\tfrac{1}{2}a_0+\sum_{\nu=1}^{n}(a_\nu \cos \nu x+b_\nu \sin \nu x) \cos (\nu\pi/(2n+1))$$

approaches the limit $f(x)$. Rogosinski proved that this method is not weaker than the method of the first arithmetic mean. The author shows by the example of the series $\sum(-1)^k k$ that the method is actually stronger than $(C, 1)$. [It is curious that, as shown by Rogosinski, if we replace $2n+1$ by $2n$ in (*), we get a method equivalent to $(C, 1)$.] The following result seems to be new. If $f(x)$ is absolutely continuous and if $T_n(x)$ is the semi-sum in (*), then

$$\int_{-\pi}^{\pi}|T_n(x)-f(x)|\,dx\leqq(A/(2n+1))\int_{-\pi}^{\pi}|f'(x)|\,dx,$$

where A is an absolute constant. Similar ideas are applied to interpolating polynomials. *A. Zygmund.*

Lorch, Lee. On Fejér's calculation of the Lebesgue constants. Bull. Calcutta Math. Soc. 37, 5–8 (1945). [MF 13301]

A new representation is given for the Lebesgue constants

$$L_n=(2/\pi)\int_0^{\pi/2}\{|\sin (2n+1)t|/\sin t\}dt$$

which avoids the use of the gamma function. The value obtained is

$$(*)\quad (4/\pi^2)\log (4n+2)+(2/\pi)\int_0^1 t^{-1}\sin t\,dt$$
$$-(2/\pi)\int_1^\infty t^{-1}\{2/\pi-|\sin t|\}dt+o(1/n).$$

A similar representation is given for the Borel-Lebesgue constants [Duke Math. J. 11, 459–468 (1944); these Rev. 6, 48].

By comparing the value of the constant term in (*) with that obtained by Watson [Quart. J. Math., Oxford Ser. 1, 310–318 (1930)] the author shows that

$$\int_1^\infty t^{-1}\{2/\pi-|\sin t|\}dt=-.16684074.$$

The value used for $(1/\pi)\,\mathrm{Si}\,(\pi)$ disagrees with the most recent computation [Szász, Duke Math. J. 11, 823–833 (1944); these Rev. 6, 125]. *P. Civin* (Buffalo, N. Y.).

Loo, Ching Tsün. The absolute summability of power series. Duke Math. J. 12, 373–380 (1945). [MF 12608]

A theorem of Hardy and Littlewood [Math. Z. 34, 620–633 (1932)] states that, if $f(z)=\sum_0^\infty a_n z^n$ and $g(z)=\sum_0^\infty b_n z^n$ are regular for $|z|<1$, the series $\sum_0^\infty a_n b_n e^{ni\theta}$ is absolutely convergent if (1) $f\varepsilon L^\lambda$, (2) $g\varepsilon\mathrm{Lip}\,(k, p)$, where $0<k\leqq 1$, $pk\geqq 1$, $\lambda=p/(p+pk-1)$ and $p\leqq 2$. The theorem fails for $p>2$, and the purpose of the paper is to show that, if $p>2$ and if the condition (2) is replaced by a more stringent one, then the series $\sum_0^\infty a_n b_n e^{ni\theta}$ is absolutely summable (C, α) for $\alpha>\frac{1}{2}-1/p$. *R. Salem* (Cambridge, Mass.).

Moursund, A. F. Non-summability of the conjugate series of the Fourier series. Duke Math. J. 12, 515–518 (1945). [MF 13520]

The divergence of the Cauchy integral

$$\lim_{\epsilon\to 0} -(2\pi)^{-1}\int_\epsilon^\pi \{f(x_0+t)-f(x_0-t)\}\cot \tfrac{1}{2}tdt$$

to $+\infty$ or $-\infty$ is a necessary and sufficient condition for the divergence of the series conjugate to the Fourier series of $f(x)$ at the point x_0 to the value $+\infty$ or $-\infty$ when summed by the method of M. Riesz which is equivalent to summability (C, δ) for $\delta>0$. This parallels the work of Prasad [Ann. of Math. (2) 33, 771–772 (1932)] which gives the same result for Abel summability of the conjugate series.

P. Civin (Buffalo, N. Y.).

Zygmund, A. Smooth functions. Duke Math. J. 12, 47–76 (1945). [MF 12070]

The function $F(x)$ of the real variable x is said to be "smooth" (briefly, $F\epsilon\lambda$) at $x=x_0$ if $F(x_0+h)+F(x_0-h)-2F(x_0)=o(h)$; $F\epsilon\Lambda$ at x_0 if $F(x_0+h)+F(x_0-h)-2F(x_0)=O(h)$. If either of these conditions is satisfied uniformly in (a, b) and F is continuous there, F is said to belong to the class λ^* or Λ^*. Finally, $F\epsilon\lambda_p$ (or Λ_p) if $F\epsilon L^p$ $(p>1)$, has period 2π and satisfies

$$\left\{\int_0^{2\pi}|F(x+h)+F(x-h)-2F(x)|^p dx\right\}^{1/p}=o(h) \qquad (\text{or } O(h)).$$

Functions of class λ^* arise naturally in the Riemann theory of trigonometric series. The author shows, by examples drawn from the theories of conjugate functions, approximation by trigonometric polynomials and fractional integrals, that the class Λ^* is, in such applications, akin to but more natural than the class Lip 1; that λ^* corresponds similarly to lip 1 (the class of functions with continuous derivatives); and Λ_p, λ_p, to the Hardy-Littlewood classes Lip$(1, p)$, lip$(1, p)$. Rajchmann [Prace Mat.-Fiz. 30, 19–88 (1919)] and Zalcwasser have shown that, if $F(x)$ is continuous and smooth in (a, b), then the set of points E at which $F'(x)$ exists finitely is dense in (a, b) and, in fact, of the power of the continuum in every subinterval. The author shows that $F'(x)$ has in E the Darboux property of taking all values intermediate between $F'(a)$ and $F'(b)$ (assuming that F is differentiable at a and b).

The following are some of the results obtained. (a) Conjugate functions. Privaloff [Bull. Soc. Math. France 44, 100–103 (1916)] proved that, if $f\epsilon$Lip α (or lip α), where $0<\alpha<1$, then so does the conjugate function f^*. This property fails for $\alpha=0$ or 1. The author proves that, if f is integrable and periodic, then (i) if f is continuous on an interval I the conjugate function f^* has the Darboux property in the subset of I for which it exists; (ii) if $f\epsilon$lip 1 (or Lip 1) in I then $f^*\epsilon\lambda^*$ (or Λ^*) in every interval interior to I. He also proves that, if f is periodic and satisfies any of λ^*, Λ^*, λ_p or Λ_p, then so does f^*. (b) Approximations. If $E_n(F)=\min_{T_n}\max|F(x)-T_n(x)|$ (the best approximation to $F(x)$ by trigonometric polynomials T_n of order n), then $E_n(F)=O(n^{-1})$ or $o(n^{-1})$ if and only if $F\epsilon\Lambda^*$ or λ^*. (c) Fractional integrals. The author proves, in extension of results of Hardy and Littlewood [Math. Z. 27, 565–606 (1928)] that, if $f\epsilon$Lip α (or lip α) then $f_{1-\alpha}\epsilon\Lambda^*$ (or λ^*). He concludes with a necessary and sufficient condition that a harmonic function should be the Poisson integral of a function of class Λ^* (or λ^*).

U. S. Haslam-Jones (Oxford).

Hsü, Hai-Tsin. The strong summability of double Fourier series. Bull. Amer. Math. Soc. 51, 700–713 (1945). [MF 13607]

It is well known that, if $s_{m,n}(x, y)$ are the partial sums of the Fourier series of a function $f(x, y)\epsilon L^p$, $p>1$, then

$$(m+1)^{-1}(n+1)^{-1}\sum_{\mu,\nu=0}^{m,n} s_{\mu,\nu}(x, y)\to f(x, y), \quad m, n\to\infty,$$

at almost every (x, y) [see A. Zygmund, Fund. Math. 23, 143–149 (1934)]. The author now shows that, if $f(x, y)\epsilon L^p$, $p>1$, and if k is any positive number, then

$$(m+1)^{-1}(n+1)^{-1}\sum_{\mu,\nu=0}^{m,n}|s_{\mu\nu}-f|^k\to 0$$

as $m, n\to\infty$.

A. Zygmund (Philadelphia, Pa.).

Bang, Thøger. On splitting the Fourier series of an almost periodic function. Mat. Tidsskr. B. 1941, 53–58 (1941). (Danish) [MF 13583]

The author's problem is the following. Under what conditions on the almost periodic function f does there exist a decomposition $f=f_1+f_2$ such that f_1, f_2 are almost periodic and f_1 has only positive, f_2 only negative Fourier exponents? Bohr's original result has been improved by Petersen [Mat. Tidsskr. B. 1937, 86–88 (1937)] and J. Favard [Leçons sur les Fonctions Presque-périodiques, Gauthier-Villars, Paris, 1933, p. 159]. The author proves that the decomposition is possible under either of the following two equivalent conditions. (1) There exist positive α, β such that the αth derivative and a βth integral are both almost periodic. (2) There exist positive α, β such that f satisfies a uniform Lipschitz condition of order α and

$$\int_\gamma^{\gamma+T} f(x)dx=O(T^{1-\beta})$$

uniformly for all γ.

František Wolf (Berkeley, Calif.).

Følner, Erling. On sets of zeros of almost periodic functions. Mat. Tidsskr. B. 1942, 54–62 (1942). (Danish) [MF 13590]

The author deduces a necessary and sufficient condition for a set to be the set of zeros of an almost periodic function. We shall call S a fundamental set if it is periodic and is generated either by a point or a closed segment. Then the condition is that the set should be the intersection of a countable number of finite sums of fundamental sets. An interesting special class of such sets are bounded closed sets. It is shown that closed sets which contain arbitrarily large intervals cannot be sets of zeros of an almost periodic function. The results are deduced for the one-dimensional case and could be extended to any number of dimensions.

František Wolf (Berkeley, Calif.).

Følner, Erling. Remark on the definition of almost periodicity. Mat. Tidsskr. B. 1944, 24–27 (1944). (Danish) [MF 13596]

Because of a result by Bogoliubov [Ann. Chaire Phys. Math. Kieff 4, 195–205 (1939)] Jessen suggested a characterization of an almost periodic function which is weaker than, but equivalent to, the classical one. We call τ an ϵ-period of f if, for all x, $|f(x+\tau)-f(x)|\leqq\epsilon$. A set S will be called relatively dense if there exists an L such that any interval of length L contains a point of S. The classical condition for f to be almost periodic is that for any positive ϵ the set E of ϵ-periods is relatively dense. The "weaker"

condition is that there should exist ϵ-periods τ_i and a positive α such that $|\tau_r-\tau_s|\geqq\alpha$, $r\neq s$, and

$$\bar{\rho}=\limsup_{T\to\infty} n\{E\cap(-T,T)\}/T>0.$$

It is obvious that a relatively dense set satisfies these two conditions; the author proves, conversely, that the set $\{\tau_i-\tau_j\}$ is relatively dense if $\{\tau_i\}$ satisfies them. The equivalence of the two characterizations of almost periodicity follows immediately without recourse to the rather deep result of Bogoliubov. *František Wolf* (Berkeley, Calif.).

Minakshisundaram, S. On the differentiated series of eigenfunctions. J. Indian Math. Soc. (N.S.) 8, 75–78 (1944). [MF 13270]

Let D be a closed region with a suitable boundary and $\omega_n(x,y)$ the eigenfunctions corresponding to the Green's function for D. Let $f(x,y)\sim\sum a_n\omega_n(x,y)$. The author obtains results of an Abelian character on the behavior of the differentiated series $\sum a_n\partial\omega_n/\partial x$ and $\sum a_n\partial\omega_n/\partial y$ in the neighborhood of a point of D. For example, if

$$\lim_{r\to 0}\int_0^{2\pi} f(x+r\cos\theta,\ y+r\sin\theta)\cos\theta d\theta=\alpha,$$

then

$$\lim_{t\to 0} t^{\frac{3}{2}}\sum a_n(\partial\omega_n/\partial x)e^{-\mu_n t}=(16\pi)^{-\frac{1}{2}}\alpha.$$

The μ_n are the eigenvalues corresponding to the ω_n.

H. Pollard (New Haven, Conn.).

Romanoff, N. P. On orthonormal systems. C. R. (Doklady) Acad. Sci. URSS (N.S.) 46, 219–221 (1945). [MF 12944]

Beginning with the basic lemma that if $k<n$ then $\sum_{d|n}\mu(n/d)f((k,d))=0$, where $f(n)$ is any function of integers, the author constructs sequences which are orthonormal and complete in abstract Hilbert space H. For example, if $f(n)$ is a function of integers and $\{f_n\}$ is a sequence in H such that $(f_m,f_n)=f((n,m))$ for all $m\geqq 1$, $n\geqq 1$, then the sequence of elements $\gamma_n=\sum_{d|n}\mu(n/d)f_d$ is orthogonal.

H. Pollard (New Haven, Conn.).

Selberg, Henrik L. Über die Darstellung der Dichtefunktion einer Verteilung durch eine Charliersche B-Reihe. Arch. Math. Naturvid. 46, no. 4, 127–138 (1943). [MF 12979]

The author discusses the possibility of representing a function $f(x)$ by a series of the form (*) $\sum c_n\Delta^n\theta(x)$, where

$$\theta(x)=\int_{-\pi}^{\pi}\varphi(t)e^{-ixt}dt.$$

He discusses properties of series of the form (*). Under some reasonable restrictions on $\theta(x)$, he shows, in particular, that a positive $f(x)$ cannot be represented by (*) if $\varphi(t)$ is analytic at $t=0$. With the same restrictions on $\theta(x)$, if $F(t)=\int_{-\infty}^{\infty}f(t)e^{ixt}dx$ converges uniformly in an interval (a,b) which has no point in common with $(-\pi,\pi)$, $f(x)$ cannot be represented by a series (*) unless $F(t)\equiv 0$ in (a,b). The Charlier B-series has the form (*) with

$$\varphi(t)=e^{-\lambda}\exp\{\tfrac{1}{2}(\lambda+\kappa)e^{it}+\tfrac{1}{2}(\lambda-\kappa)e^{-it}\}.$$

R. P. Boas, Jr. (Providence, R. I.).

Agnew, Ralph Palmer. Spans in Lebesgue and uniform spaces of translations of peak functions. Amer. J. Math. 67, 431–436 (1945). [MF 12923]

In a previous note [Bull. Amer. Math. Soc. 51, 229–233 (1945); these Rev. 6, 267] the author proved that, if $F(x)$ is a nonzero constant in a finite interval and zero outside, then the finite linear combinations of its translations are dense in $L_p(-\infty,\infty)$ for $p>1$. He extends his statement to functions of the type $F(x)=a(b-|x|)$ for $|x|\leqq b$, $F(x)=0$ for $|x|>b$ and to a more general type. *S. Bochner.*

Krein, M. On a problem of extrapolation of A. N. Kolmogoroff. C. R. (Doklady) Acad. Sci. URSS (N.S.) 46, 306–309 (1945). [MF 12950]

If $\sigma(\lambda)$ is a bounded nondecreasing function in $-\infty<\lambda<\infty$, with $\sigma(\lambda)=\sigma(\lambda-0)$ and $\sigma(-\infty)=0$, if L_σ denotes the Hilbert space of functions $\varphi(\lambda)$ for which the norm is defined by $\int_{-\infty}^{\infty}|\varphi(\lambda)|^2d\sigma(\lambda)$, and if L^+ is the subspace of L_σ consisting of linear combinations $a_1e^{it_1\lambda}+\cdots+a_ne^{it_n\lambda}$ with $t_\nu\geqq 0$, then L^+ is not dense in L_σ if and only if

$$\int_{-\infty}^{\infty}(1+\lambda^2)^{-1}\log(d\sigma/d\lambda)d\lambda>-\infty.$$

The author states that a similar condition holds if L^+ is replaced by the subspace $L^{a,b}$ for which $a\leqq t_\nu\leqq b$.

S. Bochner (Princeton, N. J.).

Beurling, Arne. Un théorème sur les fonctions bornées et uniformément continues sur l'axe réel. Acta Math. 77, 127–136 (1945). [MF 13768]

Let functions $\varphi_n(x)$, $\psi(x)$ be defined on $(-\infty,\infty)$. The author calls $\{\varphi_n\}$ strictly convergent to ψ if $\varphi_n(x)\to\psi(x)$ uniformly on every finite interval and

$$\sup_{-\infty<x<\infty}|\varphi_n(x)|\to\sup_{-\infty<x<\infty}|\psi(x)|<\infty.$$

Now let $\varphi(x)$ be bounded and uniformly continuous and consider the closed linear manifold T (in the topology of strict convergence) determined by the translations $\varphi(x+t)$, $-\infty<t<\infty$. Then, unless $\varphi\equiv 0$, T contains at least one function $e^{i\lambda x}$, λ real; at least two such functions unless $\varphi(x)\equiv e^{i\lambda x}$; and so on. This leads in a very simple way to Wiener's general Tauberian theorems. Another application is that if $\mu(x)$ is of bounded variation on $(-\infty,\infty)$ and if $\int_{-\infty}^{\infty}\varphi(x-t)d\mu(t)=0$ has a bounded uniformly continuous solution, not identically zero, then the Fourier-Stieltjes transform of $\mu(x)$ has at least one real zero.

R. P. Boas, Jr. (Providence, R. I.).

Ríos, Sixto. Lectures on the theory of the analytic continuation of Dirichlet series. Revista Acad. Ci. Madrid 37, 87 pp. (1943). (Spanish) [MF 12780]

This publication gives an account of some of the properties of Dirichlet series in the complex domain and their extension to Laplace-Stieltjes integrals. It antedates the recent publications of Mandelbrojt, and apparently Widder's book [The Laplace Transform, Princeton University Press, 1941; these Rev. 3, 232] was not available to the author, so that some recent results are not included. Among the author's own contributions are studies on overconvergence, the determination of singularities, and, in the case of Dirichlet series, the question of analytic continuation by reordering. [There seems to be a lacuna on pp. 76–77, where theorem VII and formulae [6]–[10] are omitted.]

H. Pollard (New Haven, Conn.).

Kober, H. A note on Fourier transforms. J. London Math. Soc. 19, 144–152 (1944). [MF 13636]

On replacing the kernel $\exp(-ixt)$ of the ordinary Fourier transform by $(1+itx/k)^{-k-1}$ and letting $k\to\infty$ some new formulas concerning Fourier transforms are suggested. The proofs depend on the Post-Widder inversion operator for the Laplace integral. Conditions are determined under which certain functions can be uniformly approximated by Fourier transforms. *H. Pollard* (New Haven, Conn.).

✱Ghizzetti, Aldo. Calcolo simbolico. La Trasformazione di Laplace e il Calcolo Simbolico degli Elettrotecnici. Consiglio Nazionale delle Ricerche, Monografie di Matematica Applicata, Nicola Zanichelli, Bologna, 1943. viii+331 pp.

The aim of this monograph is to supply a treatment of the Laplace transform and its applications to electric circuit theory. Accordingly, the main body of the text is divided into four parts. Part one provides an elementary discussion of the unilateral Laplace transform. In addition to derivation of the basic properties of this transform, applications are indicated to ordinary linear differential equations with constant coefficients as well as systems of such equations. A brief discussion of Bessel and theta functions is included. In the next three parts, the author applies the basic material to the solution of some of the ordinary and partial differential equations in electrotechnics. In part two, differential equations arising in lumped circuit phenomena are handled. In parts three and four, transmission lines and electric filters are discussed. A table of Laplace transforms is included in the appendix. *A. E. Heins* (Cambridge, Mass.).

Fan, Ky. Exposé sur le calcul symbolique de Heaviside. Revue Sci. (Rev. Rose Illus.) 80, 147–163 (1942). [MF 13815]

Principes généraux. Applications au calcul des intégrales définies et à l'intégration des équations différentielles. Relations avec la théorie de la composition de Volterra et applications aux équations intégrales. Étude des propriétés de certaines fonctions au moyen du calcul symbolique. *Author's summary.*

Fortet, Robert. Calcul des moments d'une fonction de répartition à partir de sa caractéristique. Bull. Sci. Math. (2) 68, 117–131 (1944). [MF 13251]

Let $F(x)$ be a distribution function,

$$\varphi(v)=\int_{-\infty}^{\infty}e^{ivx}dF(x),\quad M_n=\int_{-\infty}^{\infty}x^n dF(x).$$

The author considers the problem of determining from $\varphi(x)$ whether the M_n exist and calculating them when they do exist. His principal result is that if, in a neighborhood of $v=0$,

$$\varphi(v)=\sum_{k=0}^{n}(iv)^k a_k/k!+v^n\omega_n(v),\qquad \lim_{v\to 0}\omega_n(v)=0,$$

then M_k exists and equals a_k for $k=1,\cdots,n$ if n is even, for $k=1,\cdots,n-1$ if k is odd. *R. P. Boas, Jr.*

Cameron, R. H. Some examples of Fourier-Wiener transforms of analytic functionals. Duke Math. J. 12, 485–488 (1945). [MF 13517]

Let C (K) be the space of continuous real valued (complex valued) functions, defined on $0\leqq t\leqq 1$, which vanish when $t=0$. A "Fourier-Wiener" transform is defined for functions defined on K: $G(y)=\int F(x+iy)dx$, where $x\epsilon C$, $y\epsilon K$ and the integration is with respect to Wiener (Brownian movement) measure. Examples of functions F are given with the property that, if $F(x)\to G(y)$, $G(y)\to F(-x)$. *J. L. Doob.*

Cameron, R. H., and Martin, W. T. Fourier-Wiener transforms of analytic functionals. Duke Math. J. 12, 489–507 (1945). [MF 13518]

[See the preceding review.] Three large classes of functions F are studied and are shown to be invariant under the Fourier-Wiener transformation, to have the property described in the preceding review and to satisfy an analogue of Plancherel's and Parseval's relations. *J. L. Doob.*

Polynomials, Polynomial Approximations

Wall, H. S. Polynomials whose zeros have negative real parts. Amer. Math. Monthly 52, 308–322 (1945). [MF 12500]

A necessary and sufficient condition for all the zeros of a real polynomial $P(z)=z^n+a_1z^{n-1}+\cdots+a_n$ to have negative real parts is that, on division of $P(z)$ into the polynomial $Q(z)=a_1z^{n-1}+a_3z^{n-3}+\cdots$, the quotient is expressible as the continued fraction

$$(*)\qquad \frac{Q(z)}{P(z)}=\frac{1}{c_1z+1+}\,\frac{1}{c_2z+}\,\frac{1}{c_3z+}\cdots\frac{1}{+c_nz},$$

with $c_1>0,\cdots,c_n>0$. The proof is based on function theory and the analytic theory of continued fractions. The relation of the theorem to the well-known Hurwitz criterion is established by means of the further theorem that the representation (*) is possible if and only if none of the following determinants is zero:

$$D_1=a_1,\quad D_2=\begin{vmatrix}a_1, & a_3\\ 1, & a_2\end{vmatrix},\quad\cdots,\quad D_n=\begin{vmatrix}a_1, & a_3, & \cdots, & a_{2n-1}\\ 1, & a_2, & \cdots, & a_{2n-2}\\ 0, & a_1, & \cdots, & a_{2n-3}\\ \cdot & \cdot & \cdot & \cdot\\ 0, & 0, & \cdots, & a_n\end{vmatrix}.$$

In fact, $c_1=D_1^{-1}$, $c_2=D_1^2/D_2$ and $c_k=D_{k-1}^2/D_{k-2}D_k$ for $k=3,\cdots,n$. If there are k negative and $n-k$ positive coefficients c_p in (*), then $P(z)$ has precisely k zeros with positive real parts and $n-k$ zeros with negative real parts. *M. Marden* (Milwaukee, Wis.).

Bilharz, Herbert. Bemerkung zu einem Satze von Hurwitz. Z. Angew. Math. Mech. 24, 77–82 (1944). [MF 13193]

Hurwitz's well-known criterion for all the zeros of a polynomial $f(z)=a_0+a_1z+\cdots+a_nz^n$, with real coefficients a_k, to have negative real parts is extended in the present paper to polynomials $f(z)$ with complex coefficients $a_k=a_k'+ia_k''$. Let $D_1, D_2, \cdots$ be the even-ordered determinants

$$D_1=\begin{vmatrix}a_0' & -a_0''\\ a_1'' & a_1'\end{vmatrix},\quad D_2=\begin{vmatrix}a_0' & -a_0'' & 0 & 0\\ a_1'' & a_1' & a_0' & -a_0''\\ -a_2' & a_2'' & a_1'' & a_1'\\ -a_3'' & -a_3' & -a_2' & a_2''\end{vmatrix},\cdots,$$

where $a_k'=a_k''=0$ if $k>n$. Then all the zeros of $f(z)$ have negative real parts if and only if all the determinants D_k are positive. This is established by use of the lemma which was proved by I. Schur [Z. Angew. Math. Mech. 1, 307–311 (1921)] in connection with his proof of the Hurwitz criterion. For the computation of the determinants D_k an

algorithm is introduced similar to that used by Routh in the case of real a_k. *M. Marden* (Milwaukee, Wis.).

Sherman, S., DiPaola, J., and Frissel, H. F. The simplification of flutter calculations by use of an extended form of the Routh-Hurwitz discriminant. J. Aeronaut. Sci. 12, 385–392 (1945). [MF 13699]

The Routh-Hurwitz criterion for a real polynomial to have only zeros with negative real parts is extended to polynomials with complex coefficients. Among the extensions, which are all stated without proof, are the following (reviewer's formulation). (1) For $m=1, 2, \cdots, n-1$ and $k=0, 1, \cdots, n-m$, let

$$q_{m,k}=\Re(p_{m-1,1})p_{m-1,k+1}-\tfrac{1}{2}\{p_{m-1,k+2}+(-1)^{k+1}\bar{p}_{m-1,k+2}\},$$
$$p_{m,k}=q_{m,k}/q_{m,0}.$$

Then the polynomial $f(z)=z^n+p_{0,1}z^{n-1}+\cdots+p_{0,n}$ has only zeros with negative real parts if and only if $\Re(p_{m,1})>0$ for $m=0, 1, \cdots, n-1$. (2) Form the Sturm sequence $\{S_k(x)\}$ of which $S_0(x)=\Re(f(x))$ and $S_1(x)=\Im(f(x))$, where $x=\Re(z)$, are the first and second members. Let V_∞ and $V_{-\infty}$ be the number of variations in sign in the sequences $S_k(+\infty)$ and $S_k(-\infty)$, respectively. Then if $V_{-\infty}-V_\infty=n$, all the zeros of $f(z)$ have negative imaginary parts. Most of the paper is devoted to a discussion of computations based upon the theorems and to specific examples. Proofs will be given later. *M. Marden* (Milwaukee, Wis.).

Batschelet, Eduard. Über die absoluten Beträge der Wurzeln algebraischer Gleichungen. Acta Math. 76, 253–260 (1945). [MF 13202]

Ostrowski proved [Acta Math. 72, 99–155 (1940), in particular, p. 145; these Rev. 1, 323] that, if for a polynomial

$$A_0x^n+A_1x^{n-1}+\cdots+A_{n-1}x+A_n, \qquad A_0\neq 0,$$

the zeros, arranged according to increasing modulus, are denoted by $x_1, \cdots, x_n$, then on variation of only the arguments of the A_j the inequality $0<a_k\leqq|x_k|\leqq b_k$ holds with $b_k/a_k\leqq C_n=0.73\ (n+1)^2$. Concerning the smallest number c_n which may replace C_n in this theorem ($1<c_n\leqq C_n$), Ostrowski proved that $\liminf (c_n/n)\geqq 4/\pi$. The author proved in his thesis [Basel, 1944] that $\liminf (c_n/n)\geqq 1/\log 2$; he now proves that $c_n>(n^2/4)$ by showing that this property holds for certain equations in which n is odd and $A_{n-k}=\pm A_k$ for $k=0, \cdots, (n+1)/2$. *M. Marden* (Milwaukee, Wis.).

Hibbert, Lucien. Résolution des équations $z^n=z-a$. Bull. Sci. Math. (2) 65, 21–50 (1941). [MF 13268]

An iterative method is given for obtaining all the roots of the equation $z^n=z-a$, where n is a positive integer exceeding one and a is an arbitrary complex number. The analysis is based on the study of the loci $\arg\{(z-a)/z^n\}$ =constant and on the iteration of branches of $(z-a)^{1/n}$ for z in specified regions of the plane. Special cases ($n=3, 4, 5$) are examined. *M. H. Heins* (Providence, R. I.).

Lipka, Stephan. Über die Anzahl der Nullstellen von T-Polynomen. Monatsh. Math. Phys. 51, 173–178 (1944). [MF 12482]

A set $\varphi_0(x), \cdots, \varphi_n(x)$ of continuous real functions defined in the closed interval (a, b) forms a Chebyshev system of the nth order if every linear combination

$$P_n(x)=\sum_{\nu=0}^{n} a_\nu\varphi_\nu(x),$$

with real coefficients not all of which are zero, has at most n distinct roots in (a, b). It is well known that a necessary and sufficient condition for this is that for any $n+1$ distinct points $x_0, \cdots, x_n$ of (a, b) the determinant $|\varphi_i(x_j)|_{i,j=0}^{n}$ does not vanish.

The author shows that, for the system to be such that $P_n(x)$ has at most k distinct zeros in (a, b), it is necessary and sufficient that the rank of the matrix $\|\varphi_i(x_j)\|_{i,j=0}^{n}$ is $k+1$. He shows also that, if this condition is satisfied and if s denotes the number of zeros of $P_n(x)$ at which $P_n(x)$ undergoes a change of sign and if d is the number of the remaining zeros of $P_n(x)$, then $s+2d\leqq k$. [Cf. Dickinson, Quart. J. Math., Oxford Ser. 10, 277–282 (1939); these Rev. 1, 143.] *A. C. Offord* (Newcastle-on-Tyne).

Geronimus, J. On polynomials orthogonal on the circle, on trigonometric moment-problem and on allied Carathéodory and Schur functions. Rec. Math. [Mat. Sbornik] N.S. 15(57), 99–130 (1944). (Russian. English summary) [MF 12283]

[A preliminary report appeared in C. R. (Doklady) Acad. Sci. URSS (N.S.) 39, 291–295 (1943); these Rev. 6, 62.] The author deals with a system of polynomials $\{P_n(e^{i\theta})\}$ orthogonal on the unit circle with respect to a distribution function $\sigma(\theta)$ whose moments are

$$c_n=\int_0^{2\pi} e^{-in\theta}d\sigma(\theta).$$

He associates with $\sigma(\theta)$ the functions

$$F(z)=(2\pi c_0)^{-1}\int_0^{2\pi}\frac{e^{i\theta}+z}{e^{i\theta}-z}\,d\sigma(\theta), \quad f(z)=z^{-1}\frac{F(z)-1}{F(z)+1},$$

which he calls the Carathéodory and Schur functions, respectively. A fundamental role is played by a sequence $\{a_n\}_0^\infty$ defined in terms of the sequence $\{c_n\}_0^\infty$ by the determinants

$$(-1)^n a_n=|c_{i-k+1}|_0^n/|c_{i-k}|_0^n.$$

The author proves many results concerning the polynomials $\{P_n(z)\}$, certain related polynomials and the functions $F(z)$ and $f(z)$. The theorems are too long to quote in full. The following results may give some idea of their scope. (1) If $0<\{|a_n|\}_0^\infty<1$, then the functions $F(z)$ and $f(z)$ can be expanded in continued fractions which converge uniformly for $|z|\leqq r<1$. (2) If $\sum_{n=0}^\infty|a_n|<\infty$, then the polynomials $P_n(z)$ are of the form $z^n\{\bar{\pi}(1/z)+\epsilon_n\}$, where $\pi(z)$ is an analytic function regular for $|z|<1$ and

$$\epsilon_n=O\Big(\sum_{k=n}^\infty|a_k|\Big).$$

In this case $\sigma(\theta)$ is absolutely continuous and its derivative is positive and continuous in $0\leqq\theta\leqq 2\pi$. (3) In order that the function $f(z)=\sum_{n=0}^\infty\alpha_n z^n$, $|z|<1$, should be a Schur function it is necessary and sufficient that $|\alpha_0|<1$ and that each point $\{\alpha_n\}_1^\infty$ should lie within a circle of center B_n and radius ρ_n, where B_n and ρ_n are given in terms of the coefficients α_n. *A. C. Offord* (Newcastle-on-Tyne).

Geronimus, J. On some distribution functions connected with systems of polynomials. C. R. (Doklady) Acad. Sci. URSS (N.S.) 44, 355–359 (1944). [MF 12567]

Consider a system of polynomials $\{P_n(z)\}_0^\infty$, all of whose zeros lie in the segment $[-1, 1]$. Let numbers $\alpha_{n,k}$ be defined by

$$zP_n(z)=\sum_{k=0}^{n+1}\alpha_{n,k}P_k(z).$$

If $\psi_n(x)$ is a step-function with positive jump k/n at each root of multiplicity k of $P_n(x)$ and if the sequence $\psi_n(x)$ converges on a dense set to a limit function $\psi(x)$, then the author calls $\psi(x)$ the distribution function of the zeros of the system $\{P_n(z)\}_1^\infty$. He first finds a necessary and sufficient condition for the existence of this limit. If $A_n=\|\alpha_{i,j}\|_{i,j=0}^{n-1}$ ($\alpha_{i,j}=0$; $j>i+1$) and $A_n^k=\|\alpha_{i,j}^{(k,n)}\|_{i,j=0}^{n-1}$ the required condition is the existence for all k of the finite limit

$$\lim_{n\to\infty}\left\{n^{-1}\sum_{i=0}^{n-1}\alpha_{i,i}^{(k,n)}\right\}.$$

If the polynomials $\{P_n(z)\}_1^\infty$ are, in addition, orthogonal on the segment $[-1,+1]$ with respect to a distribution $d\sigma(x)$, $\sigma(x)$ being a bounded nondecreasing function with infinitely many points of increase, then a necessary and sufficient condition for the equivalence of $\psi(x)$ and $\sigma(x)$ is that

$$\lim_{n\to\infty}\left\{n^{-1}\sum_{i=0}^{n-1}\alpha_{i,i}^{(k,n)}\right\}=\alpha_{0,0}^{(k,n)}$$

for all k.

Besides these two distribution functions the author also considers the Robin distribution function $\mu(x)$ associated with the closure $\bar{E}$ of the set E of all zeros of the polynomials $\{P_n(z)\}$. This is defined as a distribution of unit mass on $\bar{E}$ such that the integral $\int_{-1}^{1}\log(1/|z-x|)d\mu(x)$ is constant for $z\epsilon\bar{E}$. He gives necessary and sufficient conditions for the equivalence of $\psi(x)$ and $\mu(x)$ and for the equivalence of all three functions $\psi(x)$, $\mu(x)$ and $\sigma(x)$.

A. C. Offord (Newcastle-on-Tyne).

Keldych, M. Sur l'approximation en moyenne par polynomes des fonctions d'une variable complexe. Rec. Math. [Mat. Sbornik] N.S. **16**(58), 1–20 (1945). (French. Russian summary) [MF 13010]

Detailed proofs of results announced previously [C. R. (Doklady) Acad. Sci. URSS (N.S.) **30**, 778–780 (1941); these Rev. **3**, 114].

Schaginjan, A. Sur les polynômes extrémaux qui présentent l'approximation d'une fonction réalisant la représentation conforme d'un domaine sur un cercle. C. R. (Doklady) Acad. Sci. URSS (N.S.) **45**, 50–52 (1944). [MF 12571]

Let D be a simply connected domain in the z-plane and let z_0 be an interior point of D. In the set of all functions $\{f(z)\}$, regular in D and with $f(z_0)=1$, the integral $\iint_D|f(z)|^2dxdy$ is minimized by $\varphi'(z)$, where φ is the function that maps D onto the unit circle. If $\{f(z)\}$ is the class of all polynomials of degree n or less, with $f(z_0)=1$, there is a minimizing polynomial $P_n(z)$. It is known that, if D has the additional properties that (a) it is bounded, (b) its complement is connected, then

$$(*)\qquad \lim_{n\to\infty}\iint_D|\varphi'(z)-P_n(z)|^2dxdy=0;$$

if either (a) or (b) fails to hold, there are domains for which (*) need not be valid. The author states a number of theorems (without proof), each asserting the validity of (*) for a class of domains D for which condition (a) or (b) fails. For example, let D be simply connected, unbounded, with a connected complement, and let D lie interior to a parabola $y^2=ax$ $(a>0)$. Let $T(\rho)$ be the linear measure of the arcs of the circle $x^2+y^2=\rho^2$ lying in D. If

$$\liminf_{\rho\to\infty}\{\log\log(1/T(\rho))-\tfrac{1}{2}\log\rho\}/\log\log\rho>2,$$

then (*) holds; if

$$\limsup_{\rho\to\infty}\{\log\log(1/T(\rho))\}/\log\rho<\tfrac{1}{2},$$

there are domains D for which $P_n(z)$ does not approach $\varphi'(z)$ as $n\to\infty$. *I. M. Sheffer* (State College, Pa.).

Jackson, Dunham. The boundedness of certain sets of orthonormal polynomials in one, two, and three variables. Trans. Amer. Math. Soc. **58**, 167–183 (1945). [MF 13307]

Orthonormal polynomial sets on certain algebraic curves are examined for boundedness on closed arcs. The curves in question are of the form $y^m=x^n$ in the plane and $x=t^p$, $y=t^q$, $z=t^r$ in space (m, n, p, q, r take on certain positive integral values). This extends earlier work dealing with second degree curves [Duke Math. J. **11**, 351–365 (1944); these Rev. **6**, 62]. For certain weight functions, uniform boundedness is established by reducing the problem, essentially, to the known boundedness of the Legendre set. The case of a symmetric arc is comparatively simple; that of the nonsymmetric arc is given detailed treatment.

I. M. Sheffer (State College, Pa.).

Brenke, W. C. On generating functions of polynomial systems. Amer. Math. Monthly **52**, 297–301 (1945). [MF 12498]

This paper deals principally with polynomial sets $\{Y_n(x)\}$ defined by the formal expansion $e^tf(xt)=\sum_0^\infty Y_n(x)t^n/n!$, where $f(u)$ has the formal expansion $\sum_0^\infty b_nu^n/n!$. All sets $\{Y_n\}$ have the property $xY_n'(x)=n\{Y_n(x)-Y_{n-1}(x)\}$. Some particular sets are orthogonal polynomials on a finite interval; the question of the existence of other orthogonal Y-sets is left open. *I. M. Sheffer* (State College, Pa.).

Rees, C. J. Elliptic orthogonal polynomials. Duke Math. J. **12**, 173–187 (1945). [MF 12078]

A system of elliptic orthogonal polynomials is defined by the relations

$$\int_{-1}^{1}\phi_n(x)\phi_m(x)X^{-\frac{1}{2}}dx=\delta_{m,n},$$

where $X(x,k)=(1-x^2)(1-k^2x^2)$. The first part of the paper is taken up with the study of the moments

$$\beta_n=2\int_0^1X^{-\frac{1}{2}}x^{2n}dx;$$

the author derives a recurrence formula for β_n which leads him to a new set of orthogonal polynomials. In the second half of the paper he obtains for $\phi_n(x)$ a linear homogeneous differential equation of the second order, whose coefficients depend upon a single parameter for which he gives a recurrence relation. *A. C. Offord* (Newcastle-on-Tyne).

Sheffer, I. M. Note on Appell polynomials. Bull. Amer. Math. Soc. **51**, 739–744 (1945). [MF 13613]

A set of polynomials $\{P_n(x)\}_{n=0}^\infty$, with P_n of degree exactly n, is called an Appell set if $P_n'(x)=P_{n-1}(x)$, $n=1,2,\cdots$; or, equivalently, if (formally) there is an $A(t)=\sum_{n=0}^\infty a_nt^n$, $a_0\neq0$, with

$$A(t)e^{xt}=\sum_{n=0}^\infty t^nP_n(x);$$

$A(t)$ is called the determining function for the set. The author proves that a set $\{P_n(x)\}$ is an Appell set if and only if there exists a function $\beta(x)$, of bounded variation on $(0,\infty)$, such that the moments $n!b_n=\int_0^\infty x^nd\beta(x)$ all exist,

$b_0 \neq 0$, and

$$(*) \qquad n!P_n(x) = \int_0^\infty (x+t)^n d\beta(t).$$

In this case $A(t) = \sum b_n t^n = \int_0^\infty e^{tu} d\beta(u)$.

A set of polynomials is said to be of type zero [Sheffer, Duke Math. J. 5, 590–622 (1939); these Rev. 1, 15] if (formally) there are series

$$A(t) = \sum_{n=0}^\infty a_n t^n, \quad H(t) = \sum_{n=1}^\infty h_n t^n$$

($a_0 \neq 0$, $h_1 \neq 0$), such that

$$A(t)e^{xH(t)} = \sum_{n=0}^\infty t^n P_n(x);$$

or, equivalently, $L[P_n(x)] = P_{n-1}(x)$, where

$$L[y] = \sum_{n=1}^\infty l_n y^{(n)}(x).$$

Let $\{B_n(x)\}$ be the set $\{P_n(x)\}$ corresponding to $A(t) = 1$; $B_0(0) = 1$, $B_n(0) = 0$ for $n > 0$. Then if (*) is replaced by

$$P_n(x) = \int_0^\infty B_n(x+t) d\beta(t),$$

the author's criterion for an Appell set becomes one for a set to be a set of type zero associated with the operator L, and

$$A(L(z)) = \sum_{n=0}^\infty b_n z^n, \quad A(t) = \int_0^\infty e^{uH(t)} d\beta(t).$$

R. P. Boas, Jr. (Providence, R. I.).

Sastry, B. Seetharama. A generalization of certain properties of Laguerre polynomials. Proc. Edinburgh Math. Soc. (2) 7, 83 (1945). [MF 12351]

The analogue of Rodrigues' formula is modified by writing

$$\Pi_n(z) = e^z (d/dz)^n \{e^{-z} A_n(z)\},$$

in which the function z^n is replaced by the general polynomial $A_n(z) = \sum a_k \binom{n}{k} z^{n-k}$. The author establishes the relations

$$\sum_{s=0}^{r} (-1)^s \binom{r}{s} \frac{n!}{(n-s)!} \Pi_{n-s}(z) = (-1)^n \sum_{s=0}^{n-r} (-1)^s \binom{n-r}{s} \frac{n!}{(n-s)!} A_{n-s}(z),$$

$$\sum_{s=0}^{r} (-1)^s \binom{r}{s} \frac{n!}{(n-s)!} \Pi'_{n-s}(z) = -n \sum_{s=0}^{r-1} (-1)^s \binom{r-1}{s} \frac{(n-1)!}{(n-s-1)!} \Pi_{n-s-1}(z).$$

H. E. Bray (Houston, Tex.).

Feldheim, Ervin. Relations entre les polynomes de Jacobi, Laguerre et Hermite. Acta Math. 75, 117–138 (1943). [MF 13206]

Numerous expressions are obtained (some old, others new) relating pairs of the polynomial sets of the title. These fall into several categories. (i) Allowing a parameter present in one polynomial set to approach a limiting value (usually ∞); for example, the Laguerre set is obtained from the Jacobi set:

$$L_n^{(\alpha)}(x) = \lim_{\beta \to \infty} P_n^{(\alpha,\beta)}(1 - 2x/\beta).$$

(ii) Definite integral relations, for example,

$$P_n^{(\alpha,\beta)}(x) = \frac{1}{\Gamma(n+\alpha+\beta+1)} \int_0^\infty u^{n+\alpha+\beta} e^{-u} L_n^{(\alpha)}(\tfrac{1}{2}(1-x)u) du,$$

$n+\alpha+\beta > -1$. (iii) Series expansions, for example, for the odd Hermite polynomials,

$$H_{2n+1}(x) = 2^{2n+1}(-1)^n n! \sum_{r=0}^{n} (\tfrac{1}{2}-\alpha)_r x L_{n-r}^{(\alpha)}(x^2)/r!, \quad \alpha < \tfrac{1}{2}.$$

Here $(a)_r = \Gamma(a+r)/\Gamma(a)$. Expressions are also found for the product of two polynomials. *I. M. Sheffer.*

Special Functions

Bose, B. N. On some transformations of the generalized hypergeometric series. J. Indian Math. Soc. (N.S.) 8, 120–128 (1944). [MF 13276]

The author obtains three transformations, typical of which is an expression for

$${}_4F_3(\alpha_1, \alpha_2, \alpha_3, \alpha_4; \rho_1, \alpha_1+2\alpha_4-\rho_1, \tfrac{1}{2}(\alpha_2+\alpha_3+1))$$

in terms of two well poised ${}_9F_8$ functions. These results, derived from identities obtained by manipulation of contour integral expressions for generalized hypergeometric series, are very similar to transformations given by Bailey [Generalized Hypergeometric Series, Cambridge University Press, 1935]. *N. A. Hall* (New Hartford, Conn.).

Rainville, Earl D. The contiguous function relations for ${}_pF_q$ with applications to Bateman's $J_n^{u,v}$ and Rice's $H_n(\zeta, p, v)$. Bull. Amer. Math. Soc. 51, 714–723 (1945). [MF 13608]

Defining a hypergeometric function as contiguous when it is derived from an ${}_pF_q$ by increasing or decreasing a single parameter by unity, the author develops a set of $2p+q$ linearly independent relations between an ${}_pF_q$ and its $2p+2q$ contiguous functions. These are direct generalizations of the well-known five independent relations of Gauss for ${}_2F_1$. The general formulas are applied to obtain several new relations between $J_n^{u,v}$ and $H_n(\zeta, p, v)$ functions of different orders. *N. A. Hall* (New Hartford, Conn.).

Vallarta, Manuel Sandoval. Note on the roots of some transcendental equations. Bol. Soc. Mat. Mexicana 2, 13–14 (1945). (Spanish) [MF 12868]

The author lists (a) the positive roots of Hermite's polynomial $H_n(x)$ for $n \leqq 7$; (b) the first root of the equation

$$H_4(x) = \exp(x^2/4)(2\pi)^{\frac{1}{2}}(24)^{\frac{1}{2}} J_4(x);$$

(c) the first two roots of $\mathrm{Ci}(x) - J_0(x) = 0$; (d) the first root of $J_0(x) - xM(1, \tfrac{1}{2}, x) = 0$, where Ci is the cosine integral function and M is the confluent hypergeometric function. Roots are given to 4 and 5 significant figures.

D. H. Lehmer (Aberdeen Proving Ground, Md.).

Kibble, W. F. An extension of a theorem of Mehler's on Hermite polynomials. Proc. Cambridge Philos. Soc. 41, 12–15 (1945). [MF 12371]

Generalizing a classical formula of Mehler, the author expands the expression

$$R^{-\frac{1}{2}} \exp\left(- \sum_{i,k=1}^{n} R_{ik} x_i x_k / 2R \right),$$

where R_{ik} are the minors of a given symmetrical matrix with

determinant R, into a series of products $H_{k_1}(x_1)\cdots H_{k_n}(x_n)$; $H_k(x)$ is the kth Hermite polynomial. *G. Szegö.*

Bose, B. N. On certain integrals involving Legendre and Bessel functions. Bull. Calcutta Math. Soc. 36, 125–132 (1944). [MF 12410]

The point of departure is Cooke's formula

$$\int_0^1 P_n(1-2y^2)y^{2m+1}dy=\frac{(-)^n\{\Gamma(m+1)\}^2}{2\Gamma(m-n+1)\Gamma(m+n+2)}$$

[Proc. London Math. Soc. (2) 23, xix–xx (1924)]. From this formula expansions for

$$\int_0^\infty P_n(1-2y^2)J_m(yz)y^{1+m}dy$$

are derived by term-by-term integration. Well-known manipulations lead to the replacement in several different ways of the Bessel function in the integrand by a product of Bessel functions. A considerable number of integrals are evaluated by combination of these results with known formulae. Samples are

$$\int_0^\infty \frac{J_{2n+1}(yz)dz}{z(z^2+k^2)}=\tfrac{1}{2}(-)^n\pi k^{-2}\{I_{2n+1}(ky)-\mathbf{L}_{2n+1}(ky)\},$$

$$\int_0^\infty (t^2+z^2)^{-\frac{1}{2}}J_{2n+1}(y(t^2+z^2)^{\frac{1}{2}})dt=\tfrac{1}{2}(-)^n\pi J_{n+\frac{1}{2}}(\tfrac{1}{2}yz)J_{-n-\frac{1}{2}}(\tfrac{1}{2}yz).$$

A. Erdélyi (Edinburgh).

Sircar, H. On the integrals $\int_0^x J_k(x)dx$ and $\int_0^x J_{-k}(x)dx$. Bull. Calcutta Math. Soc. 37, 1–4 (1945). [MF 13300]

By use of the Laplace transform the author derives several formulas concerning the functions $S_k(x)=\int_0^x J_k(x)dx$. For example, if $0<k<1$ then

$$\int_0^\infty \frac{S_{-k}(x)-S_k(x)}{x}dx=\int_0^\infty \frac{\{(x^2+1)^{\frac{1}{2}}-x\}^{-k}-\{(x^2+1)^{\frac{1}{2}}-x\}^{k}}{x(x^2+1)^{\frac{1}{2}}}dx.$$

The substitution $x=\sinh\theta$ reduces this to

$$2\int_0^\infty \frac{\sinh k\theta}{\sinh\theta}d\theta=\pi\tan\tfrac{1}{2}k\pi,\qquad 0<k<1.$$

H. Pollard (New Haven, Conn.).

Wells, C. P., and Spence, R. D. The parabolic cylinder functions. J. Math. Phys. Mass. Inst. Tech. 24, 51–64 (1945). [MF 12315]

In this paper the solutions of the differential equation

$$d^2U_a(\zeta)/d\zeta^2+(\zeta^2+a)U_a(\zeta)=0$$

are investigated for real values of a. The even and odd solutions of this differential equation are denoted by ${}_eU_a$ and ${}_oU_a$, respectively; they are normalized so that ${}_eU_a(0)={}_oU_a'(0)=1$. The authors give the first five terms of the power series expansion of these solutions, their expressions in terms of the confluent hypergeometric function, integral representations, asymptotic values for large a and asymptotic expansions for large ζ. There are numerical tables and graphs of the even and odd solutions for values of ζ from 0 to 3 and for $a=\pm1, \pm2, \pm3$. The tables are to four decimals, but the accuracy is not specified; ζ increases in steps of 0.1. "For values of ζ larger than those in the tables the asymptotic series in ζ is sufficiently accurate for most calculations. When the absolute value of a is larger than those for which the functions have been tabulated the asymptotic series in a may be used to extend the tables between $\zeta=0$ and $\zeta=3$." The tables were obtained by using the power series for $\zeta<1$ and then extending this range by Milne's method of numerical integration of differential equations. An auxiliary table gives the values of the modulus and phase of $\Gamma((3+ia)/4)$ for $a=1, 2, 3, 4, 5$. "These should be useful in computing the function by means of the asymptotic series in ζ." *A. Erdélyi.*

Simonart, Fernand. Sur l'équation de Bessel. Bull. Soc. Roy. Sci. Liége 12, 207–212 (1943). [MF 13137]

It is shown that the equation $xy''+2ny'+xy=0$ has the solutions

$$(D^2+1)^{n-1}\{x^{-1}\sin x\},\quad (D^2+1)^{n-1}\{x^{-1}\cos x\},\quad n\geqq 1.$$

Power series developments are obtained for these solutions.

H. Pollard (New Haven, Conn.).

McLachlan, N. W. Hill's differential equation. Math. Gaz. 29, 68–69 (1945). [MF 12538]

The equation is

$$(1)\qquad d^2y/dt^2+\{a-2k^2\psi(\omega t)\}y=0,$$

where $\psi(x)$ is a continuous function of x, periodic with period 2π. The author gives an approximate solution under the assumptions $a\gg\omega^2\gg1$, $a\gg|2k^2\psi(\omega t)|_{\max}$. With $a=\nu^2$, $y=\exp\{\nu\int w(t)dt\}$, $\rho^2=1-2k^2\psi(\omega t)/a$, (1) transforms to the Riccati equation

$$\nu^{-1}dw/dt+w^2+\rho^2=0,$$

the solution of which is expanded in the form $\sum_{r=0}^\infty w_r\nu^{-r}$. The second approximation obtained in this way gives the reviewer's result [Ann. Physik (5) 19, 585–622 (1934)]. Reference is also made to a result of Jeffreys [Proc. London Math. Soc. (2) 23, 428–436 (1924)]. The result is applied to the Mathieu equation, for which $\psi(\omega t)=\cos 2t$.

A. Erdélyi (Edinburgh).

Goddard, L. S. On the summation of certain trigonometric series. Proc. Cambridge Philos. Soc. 41, 145–160 (1945). [MF 12846]

The series in question are

$$S_n^t(\alpha)=\alpha^2n^2\sum_{m=1}^\infty\frac{\sin^2(m\pi/\alpha)}{m^t(m^2-\alpha^2n^2)},$$

$$T_n^t(\alpha)=\alpha^2n^2\sum_{m=1}^\infty\frac{\sin^2(\pi m/\alpha)}{m^{t-2}(m^2-\alpha^2n^2)^2},$$

where $\alpha>1$ and t, n are positive integers. In the first part of the paper recurrence relations are derived by means of which values for $t>2$ can be deduced from values of S_n^1, S_n^2, T_n^1, T_n^2 and

$$\theta_k(\alpha)=\sum_{m=1}^\infty m^{-k}\sin^2(\pi m/\alpha),\qquad k=2, 3, \cdots.$$

When k is even, $\theta_k(\alpha)$ can be expressed in terms of the Bernoulli polynomial of degree k; when k is odd and α is rational ($\alpha=r/s$, r, s integers), θ_k is expressed as a finite trigonometric sum (of r terms), the coefficients of which contain derivatives of the logarithm of the gamma function. Similar expressions are also obtained for S_n^1 and T_n^1 (for a rational α), while S_n^2 and T_n^2 can be summed for arbitrary α. Tables, computed by means of the formulae so obtained, are given for $-S_n^1$, $-S_n^3$, and T_n^3 for $n=1(1)10$ and $\alpha=1.0, 1.25, 1.5, 2.0, 2.5, 3(1)8$; for $-S_n^2$, T_n^2 (which are independent of n) and $\alpha^2\theta_2(\alpha)$ in the same range of α;

and for T_n^1/n for $n=1(1)10$ and $\alpha=1, 1.25, 1.5, 2, 4, 8$. The tables are to four decimal places.

In the second part of the paper integral representations are obtained for S_n^1 and T_n^1 and asymptotic series are obtained for large α. There are also representations by infinite integrals derived by the familiar contour integration methods (calculus of residues). *A. Erdélyi* (Edinburgh).

Calculus of Variations

***Pauc, Christian. Les Méthodes Directes en Calcul des Variations et en Géométrie Différentielle.** Thèse, Université de Paris, 1941. ix+139 pp.

***Pauc, Christian. La Méthode Métrique en Calcul des Variations.** Actualités Sci. Ind., no. 885. Hermann et Cie., Paris, 1941. 81 pp.

***Pauc, Christian. Les Méthodes Directes en Géométrie Différentielle.** Actualités Sci. Ind., no. 886. Hermann et Cie., Paris, 1941. 59 pp. [paged 81–139]

[The second and third titles are the two parts of the first, issued separately.] The author presents a connected account of the results of his numerous contributions to the calculus of variations and differential geometry made since 1936 by the use of direct (metric) methods and sketches the developments of these subjects (due principally to Menger and Bouligand) by such methods. *L. M. Blumenthal.*

Choquet, Gustave. Étude différentielle des minimisantes dans les problèmes réguliers du calcul des variations. C. R. Acad. Sci. Paris **218**, 540–542 (1944). [MF 12121]

Let $f(x, y, \dot{x}, \dot{y})$ be continuous in the four variables, positively homogeneous of order one and regular. For a rectifiable arc C put $l(C)=\int_C f(x, y, \dot{x}, \dot{y})dt$ and $l(a, b)=l$(segment from a to b). The simple arc C is called normal with respect to f if $l(C_{ab})/l(a, b)\to 1$ whenever $ab\to 0$, where C_{ab} is the subarc of C from a to b; C is normal with respect to f if and only if it is normal for the Euclidean length $(f=(\dot{x}^2+\dot{y}^2)^{\frac{1}{2}})$. Every minimizing arc for f is normal and every arc normal for the Euclidean length is minimizing for a properly chosen regular f. If the lower curvature of the indicatrix of f stays larger than a positive constant K independent of x, y, and if a positive function $\alpha(x)$, defined for $x>0$, exists such that $\sum\alpha(a2^{-\nu})$ converges for a suitable positive a and

$$|f(x_1, y_1, \dot{x}, \dot{y})-f(x_2, y_2, \dot{x}, \dot{y})| < \beta(\dot{x}^2+\dot{y}^2)^{\frac{1}{2}}\alpha^2([(x_1-x_2)^2+(y_1-y_2)^2]^{\frac{1}{2}}),$$

then each minimizing arc has a continuous tangent. The total curvature of an arc of length s is at most $\sum\alpha(s2^{-\nu})$. No proofs are given. It is stated that the results can be extended to higher dimensions. [The paper of Busemann and W. Mayer [Trans. Amer. Math. Soc. **49**, 173–198 (1941); these Rev. **2**, 225], which overlaps his results, was not available to the author.] *H. Busemann.*

Abdelhay, J. On a problem in the calculus of variations. Anais Acad. Brasil Ci. **17**, 45–49 (1945). (Portuguese) [MF 12765]

The author attempts to show that an extremal arc $y=g(x)$, $x_1\leqq x\leqq x_2$, for

$$J[y]=\int_{x_1}^{x_2} f(x, y, y')dx$$

affords a minimum to this integral in the fixed end-point problem if the following conditions hold along this arc: (a) $E(x, g, g', g/x)=0$ for $x\neq 0$, (b) $xg'-g\neq 0$, (c) $E(x, g, g', \lambda)>0$ for $\lambda\neq g'$ and $\lambda\neq g/x$ $(x\neq 0)$, where $E(x, g, g', \lambda)$ is the Weierstrass excess function. The arguments advanced to establish this and other related propositions are invalidated, however, by the fact that after expressing $E(x, g, g', \lambda)$ in terms of the auxiliary function $\phi(x,y,y')=f(x,y,y')/(xy'-y)$ the author omits the term $f(x, g, g/x)$ in the expression

$$E(x, g, g', g/x)=f(x, g, g/x)+(1/x)(xg'-g)^2\phi_{y'}(x, g, g')$$

and erroneously concludes from conditions (a) and (b) that $\phi_{y'}(x, g, g')=0$ on $x_1\leqq x\leqq x_2$. *W. T. Reid.*

Radon, Johann. Ein einfacher Beweis für die Halbstetigkeit der Integrale der Variationsrechnung auf starken Extremalen. Math. Ann. **119**, 205–209 (1944). [MF 11895]

The author attempts to give simple proofs of the semi-continuity of single and double integrals of the calculus of variations in parametric form. His proof for the single integral case seems to be valid, but the one for the double integral is apparently erroneous unless he has tacitly assumed hypotheses not stated in the paper. He considers the integral

$$J_\phi=\int\int_\phi F(x_1, x_2, x_3, x_4, p_{12}, p_{13}, p_{14}, p_{34}, p_{42}, p_{23})dudv,$$

with $p_{ik}=\partial(x_i, x_k)/\partial(u, v)$ and with F positively homogeneous in the Plücker coordinates p_{ik}; that is,

$$\sum_{(ik)}\sum_{(lm)}\epsilon_{iklm}p_{ik}p_{lm}=0,$$

where

$$(i, k)=[(1, 2), (1, 3), (1, 4)], (l, m)=[(3, 4), (4, 2), (2, 3)]$$

and ϵ_{iklm} is 1 for complementary pairs and 0 otherwise. He defines the E-function as

$$E(x, \pi, p, \lambda)=F(x, p)-\sum_{(ik)}p_{ik}F_{ik}(x, \pi)+\lambda\sum_{(ik)}\sum_{(lm)}\epsilon_{iklm}p_{ik}\pi_{lm}$$

with $\sum\pi_{ik}^2=1$. A surface element (x, π) is strong if there is a λ for which $E\geqq 0$ for all p and $E=0$ only for p proportional to π. A strong surface element is said to be regular when a certain determinant formed from the $F_{iklm}(x, \pi)$ is not zero. The following example, due to McShane, shows the author's integral is not semi-continuous. Take

$$F=\Big(\sum_{(ik)}p_{ik}^2\Big)^{\frac{1}{2}}+2p_{23},$$

ϕ_0 to be $x_1=u$, $x_2=v$, $x_3=x_4=0$ with $-1\leqq u\leqq 0$, $-1\leqq v\leqq 1$, and $\pi_{12}=1$, $\pi_{lm}=0$ if $(l, m)\neq(1, 2)$. Then, for $\lambda=0$, the element given above is strong and can be seen to be regular. To show the integral is not semi-continuous it suffices to consider the sequence Φ_n: $x_1=u$, $x_2=v\cos n^2u$, $x_3=v\sin n^2u$, $x_4=0$ on $(0<u\leqq 1/n, -1/n\leqq v\leqq 1/n)$; $x_1=u$, $x_2=v$, $x_3=0$, $x_4=0$ on $(-1\leqq u\leqq 0, -1\leqq v\leqq 1)$. On the rectangle $(0\leqq u\leqq 1/n, -1/n\leqq v\leqq 1/n)$, $\int\int Fdudv\to -1$ as n increases. It is possible to map the region over which Φ_n is defined onto the rectangle $(-1\leqq u\leqq 0, -1\leqq v\leqq 1)$ in such a fashion that Φ_n converges uniformly to Φ_0. Hence the double integral is not semi-continuous at Φ_0. *H. H. Goldstine.*

Barker, Charles B., Jr. The Lagrange multiplier rule for two dependent and two independent variables. Amer. J. Math. 67, 256–276 (1945). [MF 12432]

Let $f(x, y, z_1, z_2, p_1, p_2, q_1, q_2)$ and $\phi(x, y, z_1, z_2, p_1, p_2, q_1, q_2)$ be functions of class C^{iv} for all values of their arguments and let $\bar{G}$ be a closed region, the boundary G^* of which is a simple closed curve $x=x(s)$, $y=y(s)$ in which $x'''(s)$ and $y'''(s)$ satisfy uniform Hölder conditions (s=arc length). A pair $(\bar{z}_1(x, y), \bar{z}_2(x, y))$ of class C'' on $\bar{G}$ is said to be quasi-normal with respect to the differential equation $\phi=0$ on $\bar{G}$ if (i) $\bar{\phi}=\phi(x, y, \bar{z}_1, \bar{z}_2, \bar{z}_{1x}, \bar{z}_{2x}, \bar{z}_{1y}, \bar{z}_{2y})=0$ on $\bar{G}$, (ii) $AD-BC\neq 0$ on $\bar{G}$ ($A=\bar{\phi}_{p_1}$, $B=\bar{\phi}_{p_2}$, $C=\bar{\phi}_{q_1}$, $D=\bar{\phi}_{q_2}$, $E=\bar{\phi}_{z_1}$, $F=\bar{\phi}_{z_2}$), (iii) the set where

$$\frac{\partial}{\partial y}\left\{\frac{D(E-A_x-B_y)-C(F-B_x-D_y)}{AD-BC}\right\} = \frac{\partial}{\partial x}\left\{\frac{A(F-B_x-D_y)-B(E-A_x-C_y)}{AD-BC}\right\}$$

is nowhere dense in $\bar{G}$, (iv) if ζ_1 and ζ_2 are any two functions of class C'' on $\bar{G}$ which vanish on G^* and satisfy the equation of variation $A\zeta_{1x}+B\zeta_{2x}+C\zeta_{1y}+D\zeta_{2y}+E\zeta_1+F\zeta_2=0$ on $\bar{G}$, there exists a one-parameter family $(Z_1(x, y; \mu), Z_2(x, y; \mu))$ of class C'' for $(x, y)\epsilon\bar{G}$ and $|\mu|<\mu_0$ such that (a) Z_1 and Z_2 satisfy $\phi=0$ for each μ with $|\mu|<\mu_0$, (b) Z_i coincides with $\bar{z}_i$ on G^* for each μ with $|\mu|<\mu_0$, $i=1, 2$, (c) $Z_i(x, y; 0)=\bar{z}_i(x, y)$ and $Z_{i\mu}(x, y; 0)=\zeta_i(x, y)$ on $\bar{G}$, $i=1, 2$.

The author proves the following theorem. Let the pair $(\bar{z}_1(x, y), \bar{z}_2(x, y))$, of class C^{iv} and quasi-normal with respect to $\phi=0$ on $\bar{G}$, minimize $\iint_G f(x, y, z_1, z_2, z_{1x}, z_{2x}, z_{1y}, z_{2y})dxdy$ among all pairs (z_1, z_2) of class C'' on $\bar{G}$ which satisfy $\phi=0$ on $\bar{G}$ and coincide with $(\bar{z}_1, \bar{z}_2)$ on G^*. Then there is a unique function $\lambda(x, y)$ of class C' on $\bar{G}$ such that

$$(\partial/\partial x)(\bar{f}_{p_i}-\lambda\bar{\phi}_{p_i})+(\partial/\partial y)(\bar{f}_{q_i}-\lambda\bar{\phi}_{q_i})=\bar{f}_{z_i}-\bar{\phi}_{z_i}$$

on $\bar{G}$, $i=1, 2$. *C. B. Morrey, Jr.* (Berkeley, Calif.).

Courant, R. On Plateau's problem with free boundaries. Proc. Nat. Acad. Sci. U. S. A. 31, 242–246 (1945). [MF 13289]

Plateau's problem with free boundaries, which is the problem of establishing the existence of a minimal surface M minimizing the area functional and having parts of the boundary prescribed as given Jordan curves Γ, while other "free" parts of the boundary are merely restricted to lie on prescribed boundary surfaces S, has been solved by Courant [Acta Math. 72, 51–98 (1940); these Rev. 2, 61]. There remains the question as to whether detailed information can be obtained concerning the trace T, or set of boundary points, of M on S. Examples show that, even for a simply connected minimal surface whose boundary is partly a Jordan arc and partly free on a bounded surface M, T is not necessarily a continuous curve.

To fix the ideas, the author considers only doubly connected minimal surfaces M with a prescribed S and a fixed Γ outside S; the following results are then established. (a) If S is convex, then T is a continuous curve. (b) If S is convex and there exists a cone of supporting planes of S bounding together with S a portion of space which contains Γ, then T is a rectifiable curve. The sketched proofs are to be amplified in a forthcoming book.

E. F. Beckenbach (Los Angeles, Calif.).

GEOMETRY

Tschen, Y. Why. Algebraisation of plane absolute geometry. Amer. J. Math. 67, 363–388 (1945). [MF 12917]

Bolyai's absolute geometry is most naturally based on the primitive notions of point and intermediacy. Hjelmslev [Math. Ann. 64, 449–474 (1907)] suggested a different treatment, using reflections. The resulting geometry is more general, as not only continuity but even order may be lacking. This was further developed by Bachmann [Math. Ann. 113, 424–451 (1937)], using also the auxiliary geometry of "kinematic space," whose points and planes represent rotations and glide-reflections of the absolute plane. The object of these investigations was to imbed the generalized absolute plane in a projective plane (without first imbedding it in absolute 3-space, which would remove most of the difficulty). The present paper uses the same axioms as Bachmann's, and achieves the imbedding with about the same amount of labor, though avoiding the use of kinematic space. The unusual notation $<A, B>$ is used for the line AB.

H. S. M. Coxeter (Toronto, Ont.).

Jessen, Børge. A remark on the volume of polyhedra. Mat. Tidsskr. B. 1941, 59–65 (1941). (Danish) [MF 13584]

Two polyhedra A and B are said to admit similar partitions if it is possible to find decompositions $A=A_1+\cdots+A_n$, $B=B_1+\cdots+B_n$ of A and B into polyhedra such that A_i and B_i are congruent. The polyhedra admit of similar completions if it is possible to add to A and B polyhedra admitting similar partitions so that the resulting polyhedra admit similar partitions. Suppose, then, that it is possible to prove that two polyhedra A and B have the same volume, by using only additivity of volume, invariance under congruences and the familiar formula for the volume of a prism. The author proves that A and B admit of similar completions. It follows, in particular, that these three properties of the volume do not suffice to determine the volume of all polyhedra. *W. Feller* (Ithaca, N. Y.).

Pompeiu, D. La géométrie et les imaginaires: démonstration de quelques théorèmes élémentaires. Bull. Math. Phys. Éc. Polytech. Bucarest 11, 29–34 (1940). [MF 13546]

Let $ABCD$ be a square, E an arbitrary point in its plane; then one can always construct a quadrilateral with the distances EA, EB, EC, ED. *P. Erdös.*

Racliş, Nicolas. Théorème de fermeture. Bull. École Polytech. Bucarest [Bul. Politehn. Bucureşti] 12, 5–8 (1941). [MF 13550]

A simple proof of the theorem of Pompeiu quoted in the preceding review. Some similar questions are also discussed.

P. Erdös (Stanford University, Calif.).

Gheorghiu, Şerban. Sur un théorème de M. Pompeiu. Bull. École Polytech. Bucarest [Bul. Politehn. Bucureşti] 12, 221–234 (1941). [MF 13556]

Let $A_1, \cdots, A_n$ be the vertices of a regular polygon, B an arbitrary point in its plane. Then one can always construct a polygon from the distances BA_i. Also, for $n\geqq 4$, $BA_i^2\leqq\sum_{j\neq i}BA_j^2$. *P. Erdös* (Stanford University, Calif.).

de Sz. Nagy, Béla. Sur un problème pour les polyèdres convexes dans l'espace n-dimensionnel. Bull. Soc. Math. France 69, Communications et Conférences 3–4 (1941). [MF 13245]

The author notes the existence in Euclidean n-space ($n>3$) of nonsimplicial convex polyhedra which have no diagonals. *I. Kaplansky* (Chicago, Ill.).

Labrousse, A. Problème de Castillon dans l'espace. Bull. Soc. Math. France 69, 145–172 (1941). [MF 13243]

The problem is to inscribe in a given quadric surface a polygon of p sides so that the sides shall pass through p given points in a given order. It is solved both analytically and synthetically. The author gives a detailed discussion of the synthetic solution using the rectilinear generators, real or imaginary, of the given quadric. The problem may have either two or four solutions, according as p is odd or even. *N. A. Court* (Norman, Okla.).

Thébault, V. Sur les polygones réguliers de quinze et de trente côtés. Bull. Math. Phys. Éc. Polytech. Bucarest 11, 35–39 (1940). [MF 13547]

Thébault, V. Sur l'hexagone à côtés paralleles trois par trois. Bull. École Polytech. Bucarest [Bul. Politehn. București] 13, 5–9 (1942). [MF 13560]

Thébault, Victor. Sur la géométrie du quadrilatère complet. C. R. Acad. Sci. Paris 218, 97–99 (1944). [MF 13454]

Thébault, Victor. Sur la géométrie du tétraèdre. C. R. Acad. Sci. Paris 218, 25–27 (1944). [MF 13446]

Thébault, Victor. Nouvelles analogies entre le triangle et le tétraèdre. C. R. Acad. Sci. Paris 218, 262–264 (1944). [MF 13392]

Thébault, Victor. Sphères de Taylor du tétraèdre. C. R. Acad. Sci. Paris 220, 104–105 (1945). [MF 13485]

Bouvaist, Robert, et Thébault, Victor. Applications des déterminants à la géométrie du tétraèdre. C. R. Acad. Sci. Paris 220, 32–34 (1945). [MF 13480]

Bouvaist, R. Sur les points de contact du cercle des neuf points d'un triangle avec les tangents aux trois côtés. Bull. École Polytech. Bucarest [Bul. Politehn. București] 13, 3–4 (1942). [MF 13559]

Coșnița, César. Sur les paraboles inscrites à un triangle. Bull. École Polytech. Bucarest [Bul. Politehn. București] 12, 25–36 (1941). [MF 13553]

***Valeiras, Antonio. Some elementary formulas relating to the theory of unicursal curves.** Memorias sobre Matematicas (1942–44) por Antonio Valeiras, pp. 25–31, Buenos Aires, 1944 = Publ. Inst. Mat. Univ. Nac. Litoral 5, 215–220 (1945). (Spanish) [MF 12382]

Given three points on a curve whose equations in terms of homogeneous coordinates are $x=x(t)$, $y=y(t)$, $z=z(t)$, where each function is a polynomial in t, there is a determinant whose vanishing is the condition that the three points be collinear. Those elements of the determinant which are not zero or one are coefficients in $x(t)$, $y(t)$, $z(t)$ or elementary symmetric functions of the values of t which give the points. Using, in addition to the elements mentioned, x, y, z and powers of t, determinants are used in writing the equation of a chord through two given points or of a tangent at a point and to determine all the points. *J. L. Dorroh* (Baton Rouge, La.).

Hjelmslev, Johannes. Constructions with a gauged ruler. Mat. Tidsskr. B. 1943, 21–26 (1943). (Danish) [MF 13593]

A gauged ruler is a ruler on which two points A, B have been marked. It is assumed that this ruler permits the following constructions: (1) to draw a line through two given points, (2) to lay off AB on a given ray from the origin of the ray, (3) to intercalate AB between a given point O and a given line L, that is, to find the intersection of L with a circle with center O and radius AB. If the parallel axiom holds, Hilbert's results imply that all constructions with compass and ruler can also be accomplished with a gauged ruler. This result is shown to be independent of the parallel axiom, that is, it holds in absolute geometry. *H. Busemann* (Northampton, Mass.).

Hjelmslev, Johannes. On intercalations. Mat. Tidsskr. B. 1943, 1–8 (1943). (Danish) [MF 13591]

Let p, p' be pairs of corresponding points in two projective pencils in the Euclidean plane (space). The circles (spheres) with the segments pp' as diameters belong to a linear system. The envelope of the spheres is a cyclide of Dupin. Steiner proved that, if (p, p') is an involution on a conic section, then the circles with diameters pp' belong to a linear system. The following purely projective generalization of this theorem is given: if three pairs of an involution on a conic section are conjugate in a polar system π, then every pair of the involution is conjugate in π. *H. Busemann* (Northampton, Mass.).

Jongmans, F. Sur les mouvements d'un espace à quatre dimensions. II. Acad. Roy. Belgique. Bull. Cl. Sci. (5) 28, 35–42 (1942). [MF 13647]

The author uses the results of part I [same Bull. (5) 27, 650–665 (1941); these Rev. 7, 23] to determine the general displacement in Euclidean 4-space by methods of real projective geometry. The conclusion is not expressed in a very convenient form. *H. S. M. Coxeter* (Toronto, Ont.).

Benneton, Gaston. Note sur la configuration de Kummer. Bull. Sci. Math. (2) 68, 190–192 (1944). [MF 13416]

Let the constituents of a quaternion be used as homogeneous coordinates for a point or plane in projective space. It is shown that, for an arbitrary quaternion A, the sixteen quaternions

$$\begin{array}{cccc} A, & iA, & jA, & kA, \\ Ai, & iAi, & jAi, & kAi, \\ Aj, & iAj, & jAj, & kAj, \\ Ak, & iAk, & jAk, & kAk \end{array}$$

form a 16_6 configuration. The six planes (or points) incident with a given point (or plane) are the rest of those in the same row or column of this scheme.

The second half of the paper scarcely differs from R. W. H. T. Hudson, Kummer's Quartic Surface, Cambridge University Press, 1905, pp. 28–30. *H. S. M. Coxeter.*

Bosquet, René. Sur les homographies de l'espace permutables avec une polarité uniforme. Acad. Roy. Belgique. Bull. Cl. Sci. (5) 29, 666–679 (1943). [MF 13862]

Robert, Paul. Cycliques et cyclides. Bull. Soc. Math. France 71, 198–205 (1943). [MF 13235]

T. Moutard pointed out that an anallagmatic surface is the envelope of a variable sphere which is orthogonal to a fixed sphere and the center of which describes a fixed surface, the deferent [Nouv. Ann. Math. (2) 3, 306–309 (1864)]. In this lecture the case when the deferent is a nondevelopable quadric surface is discussed "en langage de géométrie moderne, c'est-à-dire sans calculs apparents." The author obtains the results due to Moutard and Laguerre. There are no precise bibliographical references.

N. A. Court (Norman, Okla.).

Linsman, M. Sur la configuration des arcs d'ordre linéaire $n+1$ d'un S_n. Bull. Soc. Roy. Sci. Liége 10, 350–354 (1941). [MF 13069]

The order of an arc A in real projective n-space S_n is the maximum number of points which A has in common with any (linear) $(n-1)$-space. The order of a point P on A is the order of a sufficiently small neighborhood of P on A. Let A_{n+1} be an arc of order $n+1$ in S_n. From a theorem of Haupt, A_{n+1} is the sum of a finite number of arcs of order n; in particular, A_{n+1} contains only a finite number of singular points, that is, points of order $n+1$. Let A_{n+1} be differentiable everywhere. Then the singular points can be provided with multiplicities. The author proves that the sum of the multiplicities of the singular points of A_{n+1} is at most $n+1$. His proof, like the one given by the reviewer [Proc. Nat. Acad. Sci. U. S. A. 27, 181–182 (1941); Ann. of Math. (2) 46, 68–82 (1945); these Rev. 2, 299; 6, 183], consists of showing that between two consecutive singular points P_1, P_2 of A_{n+1} there is at least one singular point of the projection A_n of A_{n+1} from one of its end-points. However, instead of letting a point P run on A_{n+1} from P_1 to P_2 and studying the point in which the osculating $(n-1)$-space of P intersects A_{n+1} again, he lets n different points run closely together on the arc P_1P_2 from a neighborhood of P_1 to one of P_2 and studies the $(n+1)$th point in which the $(n-1)$-space through the n points meets A_{n+1} again. This enables him to do without an additional assumption about the sum of the multiplicities of the intersections of A_{n+1} with any $(n-1)$-space and without the theorem that the osculating $(n-1)$-spaces of A_{n+1} are continuous. Furthermore, the reviewer considered only closed curves. Both versions are based on the quoted theorem of Haupt. [Supplementing the bibliography in both Linsman's paper and his own, the reviewer wishes to remark that Haupt has even proved that any A_{n+1}, whether differentiable or not, has at most $n+1$ singular points [Monatsh. Math. Phys. 40, 1–53 (1933)].]

P. Scherk (Saskatoon, Sask.).

Fabricius-Bjerre, Fr. Some remarks on plane curves of 3d order and space curves of 4th order. Mat. Tidsskr. B. 1942, 12–20 (1942). (Danish) [MF 13586]

Expository article reviewing Juel's work and related results by Haupt, Hjelmslev, B. Segre, Scherk and others.

H. Busemann (Northampton, Mass.).

Fotino, Scarlat. Contribution à l'étude de la perspective aéronautique. Bull. École Polytech. Bucarest [Bul. Politehn. Bucureşti] 12, 17–20 (1941). [MF 13552]

The solution of several metric problems in perspectives is discussed. [A more general and complete treatment of these problems may be found in E. Müller and E. Kruppa, Die Linearen Abbildungen, Leipzig-Wien, 1923, pp. 40–51.]

E. Lukacs (Cincinnati, Ohio).

Gheorghiu, Adrian. Quelques considérations sur la perspective. Bull. École Polytech. Bucarest [Bul. Politehn. Bucureşti] 13, 10–22 (1942). [MF 13561]

The author proposes the construction of a cubical lattice as an aid in drawing perspective images. *E. Lukacs.*

Traenkle, C. A. Affine Bildumformung mittels Entzerrungsgerät. Z. Instrumentenkunde 64, 90–96 (1944). [MF 13627]

The author discusses the use of a restitutional projector to produce affine transformations of the image by shifting the negative. It is assumed that the angle of tilt is small.

E. Lukacs (Cincinnati, Ohio).

Algebraic Geometry

Wiman, A. Über den Rang von Kurven $y^2=x(x+a)(x+b)$. Acta Math. 76, 225–251 (1945). [MF 13201]

Poincaré's conjecture [J. Math. Pures Appl. (5) 7, 161–233 (1901)] that, on any given rational plane cubic C of genus 1, the totality of the rational points can be deduced from a finite number by repeated application of the tangent and chord process was proved by L. J. Mordell [Proc. Cambridge Philos. Soc. 21, 179–192 (1922)]. The minimum number r of rational points of C, from which all the others can thus be deduced, is called the rank of C. It is not known whether r is bounded or not, and all the concrete examples where r has been determined have $r\leqq 4$. Here a number of cubics of the form (I) $y^2=x(x+a)(x+b)$, where a and b are rational, are obtained, of ranks not less than 4, 5 or 6.

The cubic (I) always contains 4 trivial rational points (the point at infinity of the y-axis and the three intersections with the x-axis), but at most two of these points are independent on (I). The author shows how a and b can be chosen so that (I) contains $\rho=2, 3, 4$ additional unrelated rational points; then he proves in several numerical cases that these points and the former two are independent on (I), and so $r\geqq 2+\rho$ (the exact determination of r is an intricate problem, whose solution is not attempted for any of the examples). The proof of the independence of the $2+\rho$ rational points is based on the distribution of the rational points (x, y) of (I) in classes depending on the common factors of the numbers x, $x+a$, $x+b$ taken in pairs, and the consideration of these classes as elements of a convenient Abelian group.

The author investigates first the curves (I) having $a+b=0$. By direct verification he obtains $\rho=2$ nontrivial rational points when

$$a=-b=2\cdot3\cdot5\cdot7,\quad 2\cdot3\cdot5\cdot11,\quad 2\cdot3\cdot5\cdot7\cdot13,$$
$$2\cdot3\cdot5\cdot7\cdot17,\quad 3\cdot5\cdot7\cdot11,\quad 5\cdot11\cdot13\cdot17,$$

and $\rho=3$ with

$$a=-b=2\cdot3\cdot5\cdot7\cdot11\cdot13\cdot17.$$

Then he proves that these curves have $r\geqq 4$ and $r\geqq 5$, respectively, and conjectures that there exist curves $y^2=x(x^2-a^2)$ having $r\geqq 6$ and an infinity of them having $r\geqq 5$. By considering a more general class of curves, he obtains the curves (I) with

$$(a, b)=(10, 22),\ (16, 34),\ (32, 66),\ (14, 36),\ (22, -20),$$
$$(-12, 30),\ (-2, 100);$$
$$(a, b)=(-26, 76),\ (22, 556);$$
$$(a, b)=(46, 292);$$

and proves that they have $r \geqq 4$, $r \geqq 5$, $r \geqq 6$, respectively. He also gives a method for deducing other curves of the last type, probably an infinity of them, each of which is presumably of rank not less than 6; for these, however, a and b are very large. *B. Segre* (Manchester).

✶Valeiras, Antonio. On Diophantine analysis on cubic surfaces. Memorias sobre Matematicas (1942–44) por Antonio Valeiras, pp. 59–79. Buenos Aires, 1944. (Spanish) [MF 12386]

There is first a scanty historical sketch on the Diophantine equation (1) $x^3+y^3+z^3+t^3=0$, for which two complete rational parametric solutions of the third and fourth degree are obtained by means of well-known geometric arguments [all this can be found in H. W. Richmond, Trans. Cambridge Philos. Soc. 22, 389–403 (1920), where tables of sets of small integers satisfying (1) are also given]. There follow some rather irrelevant remarks concerning the arithmetic upon general cubic surfaces [on this subject cf. B. Segre, Bull. Amer. Math. Soc. 51, 152–161 (1945); these Rev. 6, 185; several other recent papers are quoted there], the canonical equation (2) $ABC=A'B'C'$ for these surfaces (where A, B, C, A', B', C' are linear quaternary forms), and the dual of the surface (1). It is also shown that the number of lines lying on the surface $x^n+y^n+z^n+t^n=0$ (n an integer greater than 2) is $3n^2$. It should be noticed that the equation (2) is not new, as the author thinks, but goes back to Steiner [J. Reine Angew. Math. 53, 133–141 (1857) = Gesammelte Werke, vol. 2, Berlin, 1882, pp. 651–659].

B. Segre (Manchester).

Rao, C. V. H. On a theorem on the plane cubic. Bull. Calcutta Math. Soc. 36, 7–8 (1944). [MF 11211]

In his book [Introduction to Plane Geometry, Cambridge University Press, 1943; these Rev. 4, 250] H. F. Baker proved algebraically that, if a line moves so that the pairs of its common points with three given conics belong to an involution, then the line envelops a curve of class three. The present paper gives a synthetic proof.

J. L. Dorroh (Baton Rouge, La.).

von Mezynski, Ingonda Maria. Projective description of some plane sextic curves derived from conics as base curves. Bull. Amer. Math. Soc. 51, 175–184 (1945). [MF 11835]

H. P. Pettit described a method for generating curves by using two given curves and three pencils of lines [Tôhoku Math. J. 28, 72–79 (1927)]. If the given curves are conics, the generated curve is in general an octic, but under certain specializations of the position of the pencils relative to the conics the generated curve is a sextic. The sextics produced by the special positions of the pencils studied here have deficiencies zero or one and have quadruple points and double points as singularities. *J. L. Dorroh.*

Rowe, Charles H. Couples de tétraèdres de Möbius inscrits dans une quadrique (ou une biquadrique) et circonscrits à une autre quadrique (ou une développable de classe quatre). Ann. Sci. École Norm. Sup. (3) 58, 261–283 (1941). [MF 13311]

The present paper was written at the same time as one by B. Gambier on the same subject [same Ann. (3) 56, 71–118 (1939); these Rev. 1, 80]. It obtains many of Gambier's results from a different point of view and gives some additions. The author proves first that, if two quadrics Q, Q_1 in ordinary space intersect in a skew quadrilateral, then there are two ∞^3 systems of pairs of Möbius tetrahedra inscribed in Q and circumscribed to Q_1, such that the two tetrahedra of each pair correspond in a fixed null system. Hence he considers the net of quadrics circumscribed to a given Möbius pair and the tangential net of quadrics inscribed in it, and investigates the correspondence between the two nets which associates pairs of quadrics intersecting in skew quadrilaterals. Several properties of Möbius tetrahedra inscribed and circumscribed in given quadrics or pairs of quadrics are then deduced; moreover, it is shown that, if three quadrics have a common pair of conjugate lines, their 8 points of intersection can be distributed in 4 different ways as the vertices of two tetrahedra in the Möbius position [a simpler proof of this result could be obtained by noticing that the net determined by the three quadrics contains 4 reducible quadrics, which are the common elements of the two quadratic systems formed by the cones of the net]. Finally, the author studies the possibility that the two quadrics Q, Q_1 initially considered degenerate into a cone and a conic, respectively, with special attention to the case when Q_1 is the absolute; this throws light upon some metrical properties, obtained in a previous paper [B. Gambier and C. H. Rowe, same Ann. (3) 53, 329–386 (1936)], concerning the quartics of intersection of two cylinders of revolution. *B. Segre* (Manchester).

Edge, W. L. The geometrical construction of Maschke's quartic surfaces. Proc. Edinburgh Math. Soc. (2) 7, 93–103 (1945). [MF 12353]

The author's mathematical curiosity was aroused by a reference of Burnside to a quartic surface whose homogeneous equation is $x^4+y^4+z^4+t^4+12xyzt=0$ [Theory of Groups of Finite Order, Cambridge University Press, 1911, p. 371] and which is invariant for a group of $2^4\,5!$ collineations. The quartic form on the left of the equation appears as one of a set of six associated forms in a paper by Maschke [Math. Ann. 30, 496–515 (1887)]. The quartic curve D in which this surface intersects one of the reference planes gives rise to Dyck's configuration of 12 flecnodal and 16 other bitangents which are here described. The author then studies the surface itself in relationship to Klein's space configuration, which arises from a set of six linear complexes any two of which are in involution. The fifteen pairs of directrices of this configuration are each a pair of opposite edges for three of the fifteen fundamental tetrahedra. Six different synthematic totals of five tetrahedra exhaust the thirty edges, and with each of the six totals is associated one of Maschke's quartic surfaces, $\Phi_i=0$, $i=1, 2, \cdots, 6$, where Φ_i are homogeneous forms of degree 4 whose sum vanishes. The identity $4\sum\Phi_i\Phi_j\Phi_k\Phi_l=\{\sum\Phi_i\Phi_j\}^2$ defines a quartic primal Γ whose 15 nodal lines constitute a famous configuration studied by Segre, Castelnuovo and Baker. A group of 2^6 quaternary substitutions on (x, y, z, t) corresponding to 2^4 collineations on projective three-space leaves each Φ_i invariant. The direct product of this group with the group of permutations of the six Φ_i is of order $2^6\,6!$, and has $12I=\sum\Phi_i^2$ as an invariant of degree 8 in x, y, z, t. The subgroup leaving one particular Φ_i fixed is the group of order $2^4\,5!$ with which the discussion began.

J. S. Frame (East Lansing, Mich.).

Fano, Gino. Osservazioni varie sulle superficie regolari di genere zero e bigenere uno. Univ. Nac. Tucumán. Revista A. 4, 69–79 (1944). [MF 13020]

The first example of an irrational regular algebraic surface with geometric genus zero was given by Enriques

[Mem. Soc. Ital. Sci. (3) 10, 1–81 (1896)]. It is birationally equivalent to a sextic surface having the six edges of a tetrahedron for double lines. For $\pi=1$, complete linear systems of curves without double points of genus π may include curves of larger genus as isolated curves. The equation may be reduced to the form

$$0=x_1x_2x_3x_4f(x)+a_1x_2^2x_3^2x_4^2+a_2x_3^2x_4^2x_1^2 +a_3x_4^2x_1^2x_2^2+a_4x_1^2x_2^2x_3^2,$$

in which $f(x)$ is a general quadratic quaternary form. The coefficients a may be absorbed by linear transformations.

The present paper is concerned with particular cases within the system with fewer moduli; this can happen only when virtual curves (of genus zero) become effective. Given in [3] a linear system of quadrics without base points, the ∞^2 lines of this restricted system which belong to a pencil of quadrics ("special" or "principal" lines) form a (7, 3) congruence of lines, of order 7 and class 3. The map of this congruence on the ordinary quadratic representation of the lines of ordinary space on the points of [5] is a surface of order 10. It is a particular case (with 9 moduli) of the sextic already considered. The F^{10} of [5] contains 10 pairs of mutually adjoint cubic curves, the curves of each pair having one point in common. A cubic involution in [3] appears, defined by $u_i=u_i'$, the u being the coefficients of two associated cubic adjoints. *V. Snyder.*

Levi, B. The principle of correspondence of Chasles-Cremona and the order of the ruled surface of the trisecants of a curve. Math. Notae 4, 129–136 (1944). (Spanish) [MF 11480]

If C is a twisted algebraic curve and r is an arbitrary line, there is associated with each point of r a set of secants of C which cut r; by means of this association a correspondence on r is established. The determination of the number of points which correspond to themselves gives the order of the ruled surface of the trisecants of C.

J. L. Dorroh (Baton Rouge, La.).

Apéry, Roger. La géométrie algébrique. Bull. Soc. Math. France 71, 46–66 (1943). [MF 13232]

An expository article.

Apéry, R. Sur les courbes planes unicursales qui ont au plus neuf points multiples. Bull. Soc. Roy. Sci. Liége 11, 343–347 (1942). [MF 13109]

Apéry, R. Sur les congruences linéaires de courbes gauches unicursales qui possèdent une seule courbe singulière. Bull. Soc. Roy. Sci. Liége 12, 212–217 (1943). [MF 13138]

Nollet, Louis. Sur les congruences linéaires de cubiques gauches. Bull. Soc. Roy. Sci. Liége 12, 28–41, 71–80, 146–153, 220–229, 289–293 (1943). [MF 13129]

Nollet, Louis. Sur une quartique plane homologico-harmonique. Bull. Soc. Roy. Sci. Liége 12, 637–647 (1943). [MF 13157]

Lorent, H. Contribution à l'analogie entre les cubiques planes de genre un et les biquadratiques gauches de première espèce. Bull. Soc. Roy. Sci. Liége 12, 136–140, 204–207, 508–513, 632–636, 674–680 (1943). [MF 13134]

Jongmans, F. Recherches sur les congruences linéaires de cubiques gauches. Bull. Soc. Roy. Sci. Liége 12, 279–289 (1943). [MF 13144]

Jongmans, F. Complexes linéaires ∞^{n-1} de courbes rationnelles normales dans un espace à n dimensions. Bull. Soc. Roy. Sci. Liége 12, 2–20 (1943). [MF 13127]

Jongmans, F. Étude d'un système homaloïdal de l'espace à quatre dimensions. Acad. Roy. Belgique. Bull. Cl. Sci. (5) 28, 782–793 (1942). [MF 13678]

Godeaux, Lucien. Sur les foyers des congruences de courbes algébriques. Acad. Roy. Belgique. Bull. Cl. Sci. (5) 28, 236–239 (1942). [MF 13659]

Godeaux, Lucien. Une congruence linéaire de cubiques gauches. Acad. Roy. Belgique. Bull. Cl. Sci. (5) 28, 620–625 (1942). [MF 13671]

Godeaux, Lucien. Sur quelques surfaces projectivement canoniques appartenant à des variétés de Segre. Acad. Roy. Belgique. Bull. Cl. Sci. (5) 28, 49–59 (1942). [MF 13648]

Baudoux, Roger. Sur les surfaces appartenant à la variété de Segre représentant les couples de points d'une droite et d'un plan. Bull. Soc. Roy. Sci. Liége 12, 293–297 (1943). [MF 13145]

Calvo, Dolorès. Sur la variété de Segre représentant les points de deux plans. Acad. Roy. Belgique. Bull. Cl. Sci. (5) 28, 179–192 (1942). [MF 13655]

Ledoux, Henri. Sur quelques involutions déduites de la représentation plane de la surface cubique. Bull. Soc. Roy. Sci. Liége 11, 437–442 (1942). [MF 13114]

Baudoux, Roger. Sur une involution du second ordre dont les groupes se distribuent par couples sur les droites d'un complexe linéaire. Acad. Roy. Belgique. Bull. Cl. Sci. (5) 28, 211–222 (1942). [MF 13658]

Baudoux, R. Sur quelques involutions du second ordre dont les groupes appartiennent aux rayons d'un complexe linéaire. Bull. Soc. Roy. Sci. Liége 11, 287–298 (1942). [MF 13104]

Godeaux, Lucien. Sur une involution rationnelle n'ayant qu'un nombre fini de points unis, appartenant à une surface de genre quatre. Acad. Roy. Belgique. Bull. Cl. Sci. (5) 26, 9–17 (1940). [MF 13825]

Godeaux, Lucien. Sur les points unis des involutions cycliques d'ordre p^2 appartenant à une surface algébrique. Acad. Roy. Belgique. Bull. Cl. Sci. (5) 26, 28–43, 100–110, 115–128 (1940). [MF 13826]

Godeaux, Lucien. Sur les involutions du second ordre appartenant aux surfaces intersections complètes d'hyperquadriques. Acad. Roy. Belgique. Bull. Cl. Sci. (5) 28, 751–767 (1942). [MF 13677]

Godeaux, Lucien. Sur quelques points de la théorie des involutions cycliques appartenant à une surface algébrique. Bull. Soc. Roy. Sci. Liége 12, 199–204 (1943). [MF 13136]

Godeaux, Lucien. Remarque sur une involution du second ordre appartenant à une surface du sixième ordre. Bull. Soc. Roy. Sci. Liége 12, 260–263 (1943). [MF 13140]

Godeaux, Lucien. Sur certaines surfaces du quatrième ordre contenant douze droites. Acad. Roy. Belgique. Bull. Cl. Sci. (5) 28, 716–727 (1942). [MF 13675]
Godeaux, Lucien. Sur certaines surfaces du quatrième ordre contenant douze droites. II. Acad. Roy. Belgique. Bull. Cl. Sci. (5) 28, 746–750 (1942). [MF 13676]

Godeaux, Lucien. Sur la surface du quatrième ordre contenant une conique. Bull. Soc. Roy. Sci. Liége 11, 331–335 (1942). [MF 13106]

Godeaux, Lucien. Sur la surface du quatrième ordre contenant une droite et une conique. Acad. Roy. Belgique. Bull. Cl. Sci. (5) 28, 548–561 (1942). [MF 13669]

Godeaux, Lucien. Sur les surfaces possédant une conique multiple. Bull. Soc. Roy. Sci. Liége 12, 340–353 (1943). [MF 13146]

Godeaux, L. Construction d'une surface canonique du huitième ordre. Bull. Sci. Math. (2) 68, 132–144 (1944). [MF 13252]

This paper considers those algebraic surfaces of [3] whose plane sections form complete canonical systems. The first example of a surface having this property was given by Enriques [Atti Accad. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (5) 23_1, 206–214, 291–297 (1914)]. In the book by Enriques and Campedelli [Lezioni sulla Teoria della Superficie Algebriche, Padova, 1932, pp. 318–319] the outlines of a possible construction are given. The details of such a construction are given here. A surface of order 8 is given in [3], having a double curve C of order 12 and four triple points at the vertices of a tetrahedron. The curve C lies on a nonruled cubic surface having four double points at the triple points of C, but does not lie on any proper quartic surface. The system of plane sections of this surface forms a complete canonical system of the surface F. It is regular and contains the invariants $p_a=p_g=4$, $p^{(1)}=9$, $P_2=13$.

V. Snyder (Ithaca, N. Y.).

Godeaux, L. Construction du système canonique d'une surface particulière. Bull. Soc. Roy. Sci. Liége 11, 504–510 (1942). [MF 13118]

Godeaux, Lucien. Variétés mixtes de Segre-Veronese. Bull. Soc. Roy. Sci. Liége 11, 74–83 (1942). [MF 13092]

Godeaux, Lucien. Variétés à trois dimensions sur lesquelles l'opération d'adjonction est périodique. Bull. Sci. Math. (2) 68, 147–152 (1944). [MF 13248]

The sextic surface of [3], having for double curve the six edges of a tetrahedron, is deprived of canonical curves but possesses a bicanonical curve of order zero. On this surface, every linear system coincides with its own adjoint; that is, the process of adjunction is of order 2. The present paper considers whether algebraic varieties of three dimensions exist with similar properties. The following results are obtained. If an algebraic variety of three dimensions, having canonical surfaces and all pluricanonical surfaces of order zero, contains a cyclic involution of prime order p with a finite number of fixed points, the image of which is a surface having no canonical curves but possessing a bicanonical surface, then p is at most equal to 5. *V. Snyder.*

Godeaux, Lucien. Sur l'indétermination de la Jacobienne. Bull. Soc. Roy. Sci. Liége 12, 672–673 (1943). [MF 13141]

Pissard, Nelly. Sur une surface du sixième ordre. Acad. Roy. Belgique. Bull. Cl. Sci. (5) 28, 93–100 (1942). [MF 13651]
Pissard, Nelly. Sur une surface du sixième ordre. II. Acad. Roy. Belgique. Bull. Cl. Sci. (5) 28, 193–197 (1942). [MF 13656]

Pissard, Nelly. Sur les surfaces à sections hyperplanes hyperelliptiques de genre trois. Acad. Roy. Belgique. Bull. Cl. Sci. (5) 28, 859–865 (1942). [MF 13683]

Nollet, Louis. Sur les surfaces appartenant au cône sextique elliptique à trois dimensions de l'espace linéaire à sept dimensions. Bull. Soc. Roy. Sci. Liége 12, 602–607 (1943). [MF 13156]

Hanquet, Marcel. Sur les surfaces du quatrième ordre touchant quatre plans suivant quatre droites. Bull. Soc. Roy. Sci. Liége 12, 66–70 (1943). [MF 13130]

***Valeiras, Antonio. On the composition of birational quadratic transformations.** Memorias sobre Matematicas (1942–44) por Antonio Valeiras, pp. 33–40. Buenos Aires, 1944. (Spanish) [MF 12383]

It is shown analytically that a quadratic transformation in the plane of the type which transforms the point (x, y, z) into (xy, x^2, yz) or of the type which transforms (x, y, z) into $(xy, y^2, yz-x^2)$ is the product of transformations of the type in which the image of (x, y, z) is (yz, xz, xy).

J. L. Dorroh (Baton Rouge, La.).

***Valeiras, Antonio. On the uniformization of elliptic quartics.** Memorias sobre Matematicas (1942–44) por Antonio Valeiras, pp. 41–49. Buenos Aires, 1944. (Spanish) [MF 12384]

The uniformization of a quartic is accomplished by the process of putting the quartic into correspondence with a cubic and then uniformizing the cubic. *J. L. Dorroh.*

Huff, Gerald B. The completion of a theorem of Kantor. Bull. Amer. Math. Soc. 50, 692–696 (1944). [MF 11278]

A linear transformation

$$(1)\quad x_0'=nx_0-\sum r_ix_i,\quad x_i'=s_ix_0-\sum \alpha_{ij}x_i,\qquad i,j=1,\cdots,n,$$

is said to be proper if there exists a plane Cremona transformation with base points A_i which transforms the system of curves of order x_0 with multiplicities x_i at A_i into that of order x_0' with multiplicities x_i' at A_i. The transformation (1) is proper if and only if: (a) the coefficients are integers; (b) $x_0^2-\sum x_i^2$ and $3x_0-\sum x_i$ are invariant under (1); (c) for every p_0, p_i such that $p_0^2-\sum p_i^2=-1$ and $3p_0-\sum p_i=1$, $p_0'=np_0-\sum r_ip_i$ has the same sign as p_0. The necessity of these conditions follows from Riemann's theorem on the factorization of plane Cremona transformations into quadratic transformations. Sufficiency is proved by use of an earlier theorem of the author [Proc. Nat. Acad. Sci. U. S. A. 29, 198–200 (1943); these Rev. 5, 73]. *R. J. Walker.*

Godeaux, L. Sur la représentation des transformations birationnelles planes. Bull. Soc. Roy. Sci. Liége 11, 268–271 (1942). [MF 13102]

Godeaux, Lucien. Sur les courbes fondamentales de seconde espèce des transformations birationnelles de l'espace. Bull. Soc. Roy. Sci. Liége 11, 428–432 (1942). [MF 13112]

Pissard, Nelly. Sur une transformation birationnelle de l'espace. Bull. Soc. Roy. Sci. Liége 11, 234–237 (1942). [MF 13101]

Calvo, Dolorès. Sur une transformation quadratique de l'espace à huit dimensions. Bull. Soc. Roy. Sci. Liége 11, 166–170 (1942). [MF 13099]

Calvo, Dolorès. Représentation de quelques transformations birationnelles de l'espace. Bull. Soc. Roy. Sci. Liége 11, 532–547 (1942). [MF 13123]

Calvo, D. Sur la représentation d'une transformation birationnelle de l'espace. Bull. Soc. Roy. Sci. Liége 12, 407–415 (1943). [MF 13151]

Derwidué, Léon. Sur les transformations birationnelles liées à un faisceau de Halphen. Bull. Soc. Roy. Sci. Liége 12, 276–279 (1943). [MF 13143]

Dethier, G. Sur une transformation birationnelle de Jonquières de l'espace. Bull. Soc. Roy. Sci. Liége 12, 20–28 (1943). [MF 13128]

Dethier, G. Sur certaines transformations birationnelles de Jonquières de l'espace. Bull. Soc. Roy. Sci. Liége 12, 141–146 (1943). [MF 13135]

Jongmans, F. Transformations birationnelles et complexes linéaires de courbes rationnelles normales. Bull. Soc. Roy. Sci. Liége 11, 272–287 (1942). [MF 13103]

Ledoux, Henri. Sur une transformation birationnelle involutive d'ordre onze de l'espace. Bull. Soc. Roy. Sci. Liége 11, 83–91 (1942). [MF 13093]

Ledoux, Henri. Sur une transformation birationnelle involutive du treizième ordre de l'espace. Acad. Roy. Belgique. Bull. Cl. Sci. (5) 28, 101–117 (1942). [MF 13652]

Ledoux, Henri. Sur une transformation birationnelle involutive d'ordre $4(n_1+n_2)+5$ de l'espace. Acad. Roy. Belgique. Bull. Cl. Sci. (5) 28, 198–210 (1942). [MF 13657]

Ledoux, Henri. Sur une transformation birationnelle involutive d'ordre $4n\nu+3$ de l'espace. Bull. Soc. Roy. Sci. Liége 11, 91–96 (1942). [MF 13094]

Raskin, Jacques. Sur quelques transformations birationnelles involutives de l'espace. Bull. Soc. Roy. Sci. Liége 12, 81–88, 616–623 (1943). [MF 13131]

Differential Geometry

Bouligand, Georges. Sur les limites généralisées et sur leur utilisation en géométrie. Revue Sci. (Rev. Rose Illus.) 79, 51–56 (1941). [MF 12856]

In analogy to the methods of summation in calculus, "optional limits" are introduced into differential geometry. The following examples are typical. (1) Let $T: X=X(x,y,y')$, $Y=Y(x,y,y')$ be a contact transformation. Thus X and Y have continuous first derivatives satisfying the equation $Y_{y'}X_{y'}^{-1}=(Y_x+Y_yy')(X_x+X_yy')^{-1}$. If we assume only that the function $\varphi(x)$ has a continuous first derivative, then T transforms the curve $C: y=\varphi(x)$ into a Jordan continuum J which need not be rectifiable. However, J can be given an "optional tangent" at each point, namely the straight line with the slope $Y_{y'}X_{y'}^{-1}$. This definition enables the author to state the following theorem. Let C_n be a curve with continuous curvature and let J_n be the T-transform of $C_n(n=1,2,\cdots)$. If the (first order) line elements of C_n converge uniformly to those of C, then those of the J_n also converge uniformly to the optional line elements of J. (2) The asymptotic lines of a ruled surface are originally defined by a differential equation of the second order. Its integration leads to an integral equation involving only first derivatives. By means of this equation optional asymptotic lines may be introduced on a ruled surface which is defined through functions that have only continuous first derivatives. Again a theorem analogous to (1) may be stated. (3) The author points out that a certain caution is needed in the use of the optional concepts. Let $\varphi(x,y)$ be a continuous function of x and y. The curves representing the integrals of the differential equation $y'=\varphi(x,y)$ have a unique paratingent at each point but not necessarily a curvature. The transformation T maps them in general on nonrectifiable curves that have nothing to do with the integrals of the continuous field that we obtain by subjecting the line elements (x,y,φ) to T. *P. Scherk.*

Bouligand, Georges. Sur deux principes s'attachant à la séparation des suites pour l'examen du rôle des hypothèses en géométrie infinitésimale. Revue Sci. (Rev. Rose Illus.) 79, 246–249 (1941). [MF 12863]

Two principles connected with the "optional limits" introduced in the paper reviewed above are proposed and illustrated. (1) The "principle of concomitance": the optional limits in question are derivatives or differential operators. Their definitions are to be extended in such a way that identities between them which would hold under narrower assumptions remain valid. This imposes a restriction on the choice of the extensions and may even determine them uniquely. One example may be sufficient: let $\mathfrak{u}=\mathfrak{u}(t)$ and $\mathfrak{v}=\mathfrak{v}(t)$ be two vector functions of t whose scalar product $\mathfrak{u}\mathfrak{v}$ is constant. Then any concomitant definition of their derivatives $\mathfrak{u}'$, $\mathfrak{v}'$ must satisfy the condition $\mathfrak{u}'\mathfrak{v}+\mathfrak{u}\mathfrak{v}'=0$. (2) The "principle of exclusion": in the determination of an optional limit certain exceptional sequences are excluded, thus assuring again the existence of a limit and the validity of the relations of classical differential geometry. Example: let $X=X(x,y,z)$, $Y=Y(x,y,z)$ and $\phi(x)$ have continuous first derivatives, and let $T\equiv X=X(x,y,y')$, $Y=Y(x,y,y')$ be a contact transformation; T transforms the curve $y=\phi(x)$ into a Jordan continuum J. If $\phi(x)$ has a second derivative, J has a tangent whose slope

$$\frac{dY}{dX}=\frac{Y_x+Y_yy'+Y_{y'}y''}{X_x+X_yy'+X_{y'}y''}$$

is independent of y''. In the general case, J need not be rectifiable. However, the independence of dY/dX of y'' can be imitated thus: from among the sequences $\Delta x\to 0$ which determine y'' uniquely we exclude those for which the numerator or the denominator of the above expression vanishes. This exclusion leads back to the optional tangent of J defined in the preceding review. *P. Scherk.*

Bouligand, Georges. Principe de concomitance et limites généralisées. Revue Sci. (Rev. Rose Illus.) 81, 24–26 (1943). [MF 13809]

A continuation of the developments discussed in the two preceding reviews.

Tietze, Heinrich. Über Frenetsche Formeln, Poinsotsche Bewegungen und Gramsche Determinanten. J. Reine Angew. Math. 186, 116–128 (1944). [MF 13405]

This paper is largely expository. It gives a treatment, in a Euclidean space of n dimensions, of the kinematics of a rigid body about a fixed point and derives the Frenet formulas for a skew curve. Conditions on the differentiability of the functions under consideration are carefully stated. *S. Chern* (Princeton, N. J.).

Vigodsky, M. Sur les courbes fermées à indicatrice des tangentes donnée. Rec. Math. [Mat. Sbornik] N.S. 16(58), 73–80 (1945). (Russian. French summary) [MF 13014]

By the tangent indicatrix L of a curve λ is meant the locus of endpoints of vectors whose origin is a fixed point and which are congruent to the unit tangent vectors of λ. If λ is closed and has a continuously turning tangent, L is closed; it is easy to see that in this case L cannot be contained in the interior of any hemisphere and that, unless λ is plane, L must have points on both sides of every great circle of the unit sphere on which it is situated. The paper is devoted to the proof that this condition is also sufficient, that is, that for every closed spherical curve L which is not contained in a hemisphere there exists a closed curve of which L is the tangent indicatrix. The proof is based on several interesting auxiliary propositions, some of which deal with a finite number of points on the surface of a sphere, which are not contained in any hemisphere; one is that if a spherical curve L is not contained in a hemisphere then for every point B of the spherical surface there exist three arcs of L such that B is in the interior or on the boundary of every spherical triangle whose vertices are respectively on those arcs. *G. Y. Rainich.*

*The theorem is due to W. Fenchel [Jber. Deutsch Math. Verein. 39, 183–186 (1930)]

Piaggio, H. T. H. Expansions of coordinates of points of a plane curve in terms of s or ψ. Proc. Cambridge Philos. Soc. 41, 68–70 (1945). [MF 12378]

For a plane curve of which O is an ordinary point and which is such that the coordinates of a neighboring point P on the curve can be expanded in powers of s, the length of arc OP, or of ψ, the angle between the tangents at O and at P, the general terms of both these expansions are found. These results indicate errors in the s-series to s^6 given by Dockeray [An Elementary Treatise on Pure Mathematics, Bell, London, 1934] and in the term in s^{2r+1} given by Fowler [Elementary Differential Geometry of Plane Curves, Cambridge University Press, 1920]. *J. L. Dorroh.*

Chang, Su-Cheng. A generalization of the sextactic point of a plane curve. Duke Math. J. 12, 275–278 (1945). [MF 12598]

The author defines a $k$ic point of a plane curve C as a point at which a conic has k-point contact (contact of order $k-1$) with C. For $k=6$ this is the well-known sextactic point of C. Since the method previously used to obtain the canonical expansion of C at a sextactic point does not apply when $k>6$, a new method is devised which holds for $k=6$, 7, 8. By means of this method the canonical expansion of C is obtained at each of these three $k$ic points. *T. R. Hollcroft* (Aurora, N. Y.).

Lasley, J. W., Jr. On the equations of certain osculants. J. Elisha Mitchell Sci. Soc. 61, 48–54 (1945). [MF 12986]

Let $f(x)$, x real, be continuous, together with its derivatives of the first four orders. An osculating conic of the curve $y=f(x)$ at the point (α, β) is a conic that has, at the point (α, β), the same values for the ordinate and for the first four derivatives as the curve $y=f(x)$; in general, the osculating conic is unique and has 5-point contact with the original curve at the given point. If the value of $f^{iv}(\alpha)$ is taken as parameter, then the resultant 1-parameter family of 4-point conics (those having 4-point contact) is known as the pencil of penosculating conics at the given point. The author considers the case where a unique osculating conic exists. He derives the formulas necessary for writing down the equations for the unique k-parameter family of $(5-k)$-point conics, $k=0, 1, 2, 3, 4$, at a given point on the curve $y=f(x)$, and he derives results of which the following is typical. If $f^{iv}(\alpha)$ and $f'''(\alpha)$ are taken as parameters then the 2-parameter bundle of 3-point conics contains a unique family of 3-point parabolas, the pencil of penosculating parabolas. *M. O. Reade* (Arlington, Va.).

Bell, P. O. Metric properties of a class of quadratic differential forms. Bull. Amer. Math. Soc. 51, 563–571 (1945). [MF 12818]

Given a surface S and a general transversal surface S' of the metric normal congruence, an invariant quadratic form is characterized geometrically for the pair S, S'. By specializing S', geometric characterizations are made for various quadratic differential forms. *V. G. Grove.*

Godeaux, Lucien. Sur certaines congruences de Goursat. Bull. Soc. Roy. Sci. Liége 11, 328–331 (1942). [MF 13105]

Simon, Paul. Sur les congruences de droites. Bull. Soc. Roy. Sci. Liége 10, 176–186 (1941). [MF 13058]

Rozet, O. Note sur les congruences de droites. Bull. Soc. Roy. Sci. Liége 11, 338–343 (1942). [MF 13108]

Rozet, O. Sur les grilles hyperboliques. Bull. Soc. Roy. Sci. Liége 11, 432–436 (1942). [MF 13113]

Rozet, O. Sur la théorie des surfaces et les congruences de sphères. Bull. Soc. Roy. Sci. Liége 11, 630–634 (1942). [MF 13125]

Pissard, Nelly, et Rozet, O. Sur la théorie des surfaces et la transformation de Lie. Bull. Soc. Roy. Sci. Liége 13, 10–15 (1944). [MF 13160]

Ledoux, H. Sur l'enveloppe des quadriques de Lie d'une surface. Bull. Soc. Roy. Sci. Liége 11, 471–478 (1942). [MF 13116]

Charrueau, André. Sur la courbure et la torsion géodésique. Bull. Soc. Math. France 71, 20–26 (1943). [MF 13230]

The paper contains a pictorial representation of the normal curvature and geodesic torsion of a curve on a surface (in Euclidean three-dimensional space) in terms of the

principal curvatures and lines of curvature. Several applications are given. *A. Fialkow* (New York, N. Y.).

Vidal, Enrique. On an equivalent representation of a portion of a curved surface on a plane. Portugaliae Math. 4, 199–202 (1945). (Spanish. French summary) [MF 13333]

The element of area of a curved surface is derived through the use of a family of geodesic parallels defined on the surface. It is proved that, if the curves v^2=constant are geodesics and make a constant angle (other than a right angle) with the curves v^1=constant, then the surface is developable. A discussion is given of the relationship between the areas of noninfinitesimal regions of a curved surface and those of a developable surface tangent to it along a curve.
C. B. Allendoerfer (Haverford, Pa.).

Vidal, Enrique. Some properties of spherical curves. Revista Mat. Hisp.-Amer. (4) 4, 238–242 (1944). (Spanish) [MF 12991]

This paper contains (1) another proof of the fact that the total torsion of any closed curve on the sphere is zero, (2) a simplification of a proof by H. Geppert that this property characterizes the spheres and planes, (3) the observation that on every closed spherical curve there are at least two points whose tangents are parallel and at least two points whose principal normals are parallel.
P. Scherk (Saskatoon, Sask.).

Ostrowski, Alexander. Mathematische Miszellen. XIX. Zur integrallosen Bestimmung der Berührungstransformationen vom Range 1. XX. Ueber eine Klasse von Berührungstransformationen. XXI. Ueber eine Klasse von kanonischen Transformationen. Verh. Naturforsch. Ges. Basel 52, 35–39, 40–43, 44–48 (1941). [MF 13701]

[Note XVIII of this series appeared in Jber. Deutsch. Math. Verein. 43, 58–64 (1933).] These notes deal with some special contact transformations. Any contact transformation leads to a system $\Omega_\alpha(X_0, \cdots, X_n; x_0, \cdots, x_n)=0$, $\alpha=1, \cdots, k$, by means of which a point is made to correspond to an $(n+1-k)$-dimensional variety. Conversely, if a system of Ω's is to define a contact transformation, they must satisfy certain conditions given by Lie. The first note proves that, if the transformation is of rank 1 $(k=n)$, a necessary and sufficient condition on the Ω's is that they are not reducible simultaneously to the form $F_\alpha(X)-f_\alpha(x)=0$. In the second note the author considers the most general transformation for which $\partial X_\alpha/\partial x_\beta=0$ for $\alpha\neq\beta$. Such a contact transformation is given by $X_\alpha=\varphi_\alpha(x_\alpha+T'_{p_\alpha}(p))$, $P_\alpha=p_\alpha/\varphi_\alpha'$. The third note generalizes the second by requiring, instead of $P_\alpha dX_\alpha-p_\alpha dx_\alpha=0$, that this expression is only the differential of a function of the x's and p's. *M. S. Knebelman* (Pullman, Wash.).

Ostrowski, Alexandre. Sur les transformations réversibles d'éléments de ligne. Acta Math. 75, 151–182 (1943). [MF 13208]

By a reversible line element transformation R the author understands a transformation

$$(1)\qquad y_i=y_i(x_1, \cdots, x_n; p_1, \cdots, p_n), \quad i=1, \cdots, n,$$

$p_i=dx_i/dt$, homogeneous of degree zero in the p's and such that the elimination of p and dp/dt from (1) and from $dy_i/dt=q_i$ leaves (2) $x_i=x_i(y_1, \cdots, y_n; q_1, \cdots, q_n)$, x_i being homogeneous of degree zero in the q's. For $n=2$ a transformation R is a contact transformation; for $n>2$ if R is a contact transformation it is necessarily an extended point transformation. The important theorem for an R transformation is that the rank of the Jacobian matrix of (1) with respect to the p's and of (2) with respect to the q's is the same and is at most $n/2$. If this rank is k, any transformation R can be extended to a point transformation R^* in a space of $n+k$ dimensions; R^* is the characteristic transformation of R. The problem the author solves is that of finding all transformations R for which a given transformation R^* is characteristic. [The geometric illustrations in §57 are not very cogent since the results mentioned would hold even if (1) were not an R transformation. There is a misprint in this section: the variety corresponding to a point is of dimension k and not $n-k$.] *M. S. Knebelman.*

Kasner, Edward, and DeCicco, John. A generalized theory of contact transformations. Univ. Nac. Tucumán. Revista A. 4, 81–90 (1944). [MF 13021]

The authors prove that the only union-preserving transformations (in the whole domain of plane differential elements) are (A) the Lie group of contact transformations, (B) union-preserving transformations from differential elements of order n $(n\geqq 2)$ into lineal elements and extensions of these two sets. As in Lie's theory for (A), the transformations of (B) are determined by means of a directrix equation. Various characterizations of the transformations of (B) are given. *A. Fialkow* (New York, N. Y.).

Kasner, Edward, and DeCicco, John. Generalized transformation theory of isothermal families. Univ. Nac. Tucumán. Revista A. 4, 91–104 (1944). [MF 13022]

Kasner has previously found the total group of plane lineal-element transformations which convert every isothermal family of curves into an isothermal family. In the present paper, all admissible transformations from field elements of order n to lineal elements which preserve the isothermal character of any isothermal family of curves are found. There are seven types of these transformations in the complex plane and three types in the real plane.
A. Fialkow (New York, N. Y.).

DeCicco, John. Equilong geometry of third order differential elements. Nat. Math. Mag. 19, 276–282 (1945). [MF 12475]

Gambier, B. Système d'équations aux dérivées partielles d'ordre cinq vérifié par la surface générale de translation. Bull. Soc. Math. France 71, 1–19 (1943). [MF 13229]

The inspiration for this paper lies in a conjecture by Lie and Scheffers that all translation surfaces can be defined by a certain partial differential equation of the fourth order. The present study shows this to be incorrect. Instead, a general translation surface is the general solution of a system of two partial differential equations of the fifth order. These equations are obtained by expressing that two algebraic equations A, B of degrees four and five, respectively, have two common solutions. The coefficients of these algebraic equations are functions of partial derivatives up to the fourth and fifth orders, respectively, and are given explicitly. Translation surfaces admitting two translation nets are the solution of four equations of fifth order obtained by expressing that all roots of A are also roots of B. Surfaces admitting a single infinity of nets are the solution of three equations of fifth order obtained by expressing that A vanishes identically. In the process of proof certain known results are yielded as by-products. For example, if the

surface is algebraic the curves of the net are also. There exists an eighteen parameter family of translation surfaces having two nets and a twelve parameter family with a single infinity of nets. The case where one or both families of a net are plane curves is given special treatment.

J. L. Vanderslice (College Park, Md.).

Biran, Lutfi. Sur certaines propriétés des surfaces réglées. Rev. Fac. Sci. Univ. Istanbul (A) 8, 193–205 (1943). (French. Turkish summary) [MF 12970]

Let S be a ruled surface and let a second ruled surface S^* be defined by giving at every point of the line of striction of S a generating line of S^*. A number of theorems of the following type are proved about this situation. Consider the three conditions: (a) S and S^* have the same line of striction, (b) the angle between generating lines of S and S^* is constant, (c) the distribution parameters of S and S^* are equal. Then (a) and (b) together imply (c); (a) and (c) together imply (b); there are two surfaces S^* which satisfy (b) and (c), while one of them also satisfies (a). Similar theorems are given for developable surfaces and for triples of ruled surfaces, generated by the edges of the Blaschke trihedron of one of them.

H. Samelson.

Biran, Lutfi. Représentation géométrique des propriétés intrinsèques d'une courbe gauche. Rev. Fac. Sci. Univ. Istanbul (A) 8, 339–344 (1943). (French. Turkish summary) [MF 12969]

At the central point of a line A_1 of a ruled surface S consider the Blaschke trihedron formed by A_1, the normal to the surface, and a third line orthogonal to the other two. The instantaneous axis of rotation $A_1^{(1)}$, as the trihedron moves along the line of striction, generates another ruled surface $S^{(1)}$. The same construction can be applied to $S^{(1)}$ and yields a surface $S^{(2)}$, etc. If S is generated by the tangents to a curve C, then the system $A_1, A_1^{(1)}, \cdots, A_1^{(n+1)}$ forms a geometric representation of the intrinsic properties of C of order n in the sense that the curvature and the torsion of C and their first n derivatives can be computed from these lines. Dual vectors ($a+\epsilon b$, $\epsilon^2=0$) are used in the development.

H. Samelson (Syracuse, N. Y.).

Cotton, Émile. Intersection de deux surfaces définies par des trièdres mobiles. Bull. Soc. Math. France 69, Communications et Conférences 10–21 (1941). [MF 13247]

Let S and S^* be two surfaces in 3-space and consider the family of curves obtained by intersecting any surface congruent to S with any surface congruent to S^*. Using Darboux's method of the moving trihedron the author derives a system of differential equations for these curves in terms of the parameters on the surfaces and the Euler angles corresponding to the two trihedra at a point of one of the curves. The singularities of the system, which occur whenever the two surfaces are tangent to each other, are studied, and finally the singularities of the curves are investigated by the methods used for algebraic curves.

H. Samelson (Syracuse, N. Y.).

MacQueen, M. L. A note on hypergeodesics and canonical lines. Bull. Amer. Math. Soc. 51, 400–404 (1945). [MF 12519]

Let $l_2(k)$ be a canonical line of the second kind (not an axis of Čech) of a surface S at a point P, and let $\pi(k)$ be the harmonic conjugate of the tangent plane to S at P with respect to the focal planes of $l_2(k)$. A curve C on S through P has one of its asymptotic osculating quadrics tangent to $\pi(k)$ only if it is a member of a family of hypergeodesics. Thus two families of hypergeodesics are associated with the line $l_2(k)$. The envelope of the osculating planes of each of these families of hypergeodesics is a quadric cone with vertex at P. These cones are used to give a simple geometric construction of a canonical line of the first kind from a given line of the second kind. In particular, the construction yields interesting relationships among the second edge of Green, the first directrix of Wilczynski and the first edge of Green.

V. G. Grove (East Lansing, Mich.).

Llensa, Georges. Étude de certains systèmes triples orthogonaux. Bull. Sci. Math. (2) 65, 225–250 (1941). [MF 13262]

Darboux has introduced, by means of a differential equation, a class of 1-parameter families of surfaces (here called LD families, after Darboux and Lamé), which can be imbedded in a triply orthogonal system of surfaces. The author gives a geometrical construction of these families. Consider a surface Σ in the interior of a sphere S; introduce in S the non-Euclidean geometry of Poincaré's model. A pseudoparallel surface to Σ is obtained by shifting every point of Σ along the non-Euclidean normal to Σ (which is a circle orthogonal to S) a constant non-Euclidean distance. An LD family is now characterized as a family Σ_u of surfaces for which there exists a family of spheres S_u such that the "neighboring" surface Σ_{u+du} is obtained from Σ_u by an infinitesimal pseudoparallelism with respect to the sphere S_u. The construction of the triply orthogonal system is done purely geometrically.

Various special topics are then studied: LD families which contain a sphere (they are shown to consist entirely of spheres); Dupin's cyclides; LD families for which the family S_u is a fixed sphere; triply orthogonal systems in which two of the families of surfaces are LD; construction of such systems from initial data involving two families of associated spheres S_u and S_v and a curve which carries two orthogonal curvature bands.

H. Samelson.

Delgleize, A. Sur les transformations de Ribaucour et les surfaces isothermiques. Bull. Soc. Roy. Sci. Liége 13, 233–242 (1944). [MF 13179]

Haag, J. Lignes asymptotiques d'une surface représentée par une fonction harmonique. Bull. Sci. Math. (2) 65, 100–103 (1941). [MF 13263]

Let $Z=Q(x, y)$ be a harmonic function and S the surface defined by the function. Placing $\zeta=x+iy$, there exists an analytic function $f(\zeta)$ such that $f(\zeta)=P(x, y)+iQ(x, y)$. If $\int\sqrt{f''(\zeta)}d\zeta=U(x, y)+iV(x, y)$, the asymptotic curves on S are given by U=constant, V=constant. The functions U and V are conjugate harmonic functions, and hence the projections of the asymptotic curves of S on the (x, y)-plane form an isothermally orthogonal net. A similar discussion shows that the asymptotic curves on the surface S' defined by $P(x, y)$ are given by $U+V$=constant, $U-V$=constant. The projections of the asymptotic curves of S and S' on the (x, y)-plane intersect at an angle of 45°.

V. G. Grove (East Lansing, Mich.).

Charrueau, André. Sur les surfaces représentatives des fonctions harmoniques. Bull. Sci. Math. (2) 67, 168–176, 179–187 (1943). [MF 12638]

Let $F_1(x, y)$ and $F_2(x, y)$ be two harmonic functions, and let S_1, S_2 be the two surfaces $z_1=F_1(x, y)$ and $z_2=F_2(x, y)$. The points A_1 and A_2 of S_1 and S_2, respectively, are said

to correspond if their orthogonal projections on the (x, y)-plane are identical. This paper considers the metric differential properties of S_1 and S_2 at A_1 and A_2. A few of the simpler results may be stated as follows. The projections on the (x, y)-plane of the plane sections of S_1, S_2 by planes parallel to the (x, y)-plane form an orthogonal net. If two closed curves on S_1, S_2 have the same projection on the (x, y)-plane, they have the same area. The surfaces S_1, S_2 have equal total curvatures at A_1, A_2. Two families of surfaces (S_1'), (S_2') are found such that a surface of (S_1') corresponds to S_1, with orthogonality of linear elements, and similarly for (S_2') and S_2. Pairs of applicable surfaces are found in a simple manner from S_1 and a surface of (S_1'), S_2 and a surface of (S_2'). *V. G. Grove.*

Charrueau, André. Sur les surfaces représentatives des fonctions harmoniques. Bull. Sci. Math. (2) **68**, 193–203 (1944). [MF 13417]

This paper continues the author's discussion of pairs of surfaces represented by conjugate harmonic functions [cf. the preceding review]. Particular attention is paid to surfaces corresponding with orthogonality of linear elements. *V. G. Grove* (East Lansing, Mich.).

Vincensini, Paul. Surfaces harmoniques, congruences et représentations conformes associées. Bull. Sci. Math. (2) **68**, 60–72 (1944). [MF 13257]

Let $Z=P(x, y)$ and $Z'=Q(x, y)$ be conjugate harmonic functions and S and S' the surfaces represented by them. The function $Z_\alpha=Z\cos\alpha+Z'\sin\alpha$, where α is a real constant, is harmonic. The surface S_α defined by Z_α is called the harmonic associate of S, and α the angle of association. The following theorems serve to illustrate some relations between the properties of S and S_α. The nets formed by the projections of the asymptotic curves on S and S_α on the (x, y)-plane cut at the constant angle of $\frac{1}{2}\alpha$. The projections on the (x, y)-plane of the sections of S and S_α by a plane $Z=$constant cut at the constant angle α. Let X' be the projection on the (x, y)-plane of the point X on S, and l the line through X' parallel to the normal to S at X. The locus of l is a congruence whose middle surface is the (x, y)-plane. Conversely, given a surface S, if the lines l generate a congruence with the (x, y)-plane as its middle surface, then S is harmonic. *V. G. Grove.*

Vincensini, Paul. Les surfaces de Voss et la déformation des réseaux cinématiquement conjugués. J. Math. Pures Appl. (9) **20**, 427–440 (1941). [MF 13367]

Let S_1, S_2 be two distinct applicable surfaces and M_1, M_2 their corresponding points. Select on S_1 a one-parameter family of curves; let C be the curve of the family through M_1 and let t' be its tangent at M_1. Roll S_2 along S_1 so that M_2 moves along C. The instantaneous axis t of this motion lies in the tangent plane to S_1 at M_1. The two lines t, t' are said to form kinematically conjugate directions; the net determined by them is a kinematically conjugate net. In general such a net is not a conjugate net in the sense of Dupin. If S_1, S_2 are symmetric to a plane, such a net is also a conjugate net. There exists but one kinematically conjugate net on S_1 and S_2 which is also conjugate on both S_1 and S_2. A third surface S_3 applicable to S_1 and S_2 is to be determined such that the conjugate nets on S_3 correspond in the deformation to the kinematically conjugate nets on S_1, S_2. The three surfaces S_1, S_2, S_3 are three surfaces of a family of applicable surfaces such that in the deformation of the surfaces a conjugate net remains conjugate in the deformation. The restrictions on these surfaces furnish a new geometric definition of the surfaces of Voss: those surfaces S_1 capable of being deformed into a surface S_3 so that its kinematically conjugate nets (defined by S_1 and an applicable surface S_2) deform into conjugate nets on S_3 are the surfaces of Voss, and those surfaces are the only ones. *V. G. Grove* (East Lansing, Mich.).

Cartan, Élie. Sur une classe de surfaces apparentées aux surfaces R et aux surfaces de Jonas. Bull. Sci. Math. (2) **68**, 41–50 (1944). [MF 13255]

To a conjugate net in ordinary space correspond two equations of Laplace, a punctual one and a tangential one. Each of these equations has two differential invariants, say h, k and $\bar{h}$, $\bar{k}$. Three possibilities arise in equating these invariants: (1) $h=k$, $\bar{h}=\bar{k}$; (2) $h=\bar{k}$, $\bar{h}=k$; (3) $h=\bar{h}$, $k=\bar{k}$. The first two cases have been extensively studied in the literature. The surfaces sustaining such nets are, respectively, the surfaces of Jonas and the R-surfaces. In this paper the nets of class (3) are studied. It is proved that the corresponding surfaces, called the E-surfaces, depend on five arbitrary functions in one variable and that the characteristic curves of the differential system are the two families of curves of the net, each counted twice, and a fifth family related to the net in a simple way. [There is an obvious misprint in the differential equation of the fifth family of characteristic curves.] *S. Chern* (Princeton, N. J.).

Roşca, Radu. Sur les réseaux (M). Bull. Math. Soc. Roumaine Sci. **45**, 49–54 (1943). [MF 12755]

Let two conjugate nets N and N' be related by the Koenigs-Moutard correspondence. This paper gives in explicit form the correspondence which exists between the conjugate nets of the same index of the two sequences of Laplace transforms of N and N'. *V. G. Grove.*

Hsiung, Chuan-Chih. New geometrical characterizations of some special conjugate nets. Duke Math. J. **12**, 249–253 (1945). [MF 12596]

Let S_x be a surface and N_x a conjugate net on it. Let S_y, S_z be the surfaces generated by the Laplace transforms y and z of a generic point x of S_x. There is a quadric having second order contact with S_y (S_z) at y (z), and first order contact with S_z (S_y) at z (y). These quadrics, called associated quadrics, are used to characterize special conjugate nets. Representative theorems may be stated as follows. The tangent plane to S_x at x intersects the associated quadrics in coincident conics only if N_x has equal point invariants. The quadrics coincide only if the given net is harmonic and has equal point and equal tangential invariants. There is a composite cone in the pencil determined by the quadrics, one component of which is the tangent plane to S_x at x if and only if the net has equal point invariants. *V. G. Grove* (East Lansing, Mich.).

Hsiung, Chuan-Chih. Some invariants of certain pairs of hypersurfaces. Bull. Amer. Math. Soc. **51**, 572–582 (1945). [MF 12819]

By a method similar to that used previously [Duke Math. J. 10, 717–720 (1943); these Rev. 5, 158], the author finds invariants of pairs of hypersurfaces in a projective space of n dimensions in the cases in which the hypersurfaces have a common tangent hyperplane at corresponding points or in which the tangent hyperplanes are distinct but their intersection contains the line joining the corresponding points of

the hypersurfaces. In each case he makes projective characterizations of the invariants in terms of cross ratios.

V. G. Grove (East Lansing, Mich.).

Hsiung, Chuan-Chih. A projective invariant of a certain pair of surfaces. Duke Math. J. 12, 441–443 (1945). [MF 13511]

In a previous paper [same J. 10, 717–720 (1943); these Rev. 5, 158] the author gave a projective characterization of an invariant of a configuration composed of a pair of surfaces and a point on each, such that the tangent planes to the surfaces at those points coincide. The present paper gives a simple metric characterization of that invariant in terms of the Gaussian curvatures of the surfaces and their normal curvatures in the direction of their common tangent line.

V. G. Grove (East Lansing, Mich.).

Su, Buchin. A new invariant of intersection. Univ. Nac. Tucumán. Revista A. 4, 321–327 (1944). [MF 13031]

This paper generalizes to hypersurfaces a result of Hsiung [Bull. Amer. Math. Soc. 49, 877–880 (1943); these Rev. 5, 158]. The analytic form of the projective invariant in question is determined by use of a subclass of projective coordinate systems invariantly connected with the two hypersurfaces. Then it is interpreted geometrically in terms of $n-1$ cross-ratios formed with certain covariant lines. Specialization to the metric case leads to an expression of the invariant in terms of total curvatures.

J. L. Vanderslice (College Park, Md.).

Su, Buchin. Plane sections through an ordinary point of a hypersurface. Univ. Nac. Tucumán. Revista A. 4, 329–362 (1944). [MF 13032]

The notion of Moutard-Čech pencil (of hyperquadrics) associated with a nonasymptotic tangent t at a point O of a hypersurface V_n in S_{n+1} is defined and elaborated. Then a generalization of the cone of Kubota is obtained as locus of the projective normals of all plane sections of V_n passing through t. It is a cubic hypercone with t as double generator. A generalization of the Moutard-Čech pencil arises by considering space sections V_2 of V_n produced by linear 3-spaces through a fixed tangent plane of V_n at O. Further generalizations of the cone of Kubota result. The locus of a canonical $c(k)$ of V_2 (the first edge of Green excepted) turns out to be a fourth order cone. Consideration of a variable plane through a fixed asymptotic tangent leads to theorems on the nature of the loci swept out by certain Bompiani osculants of the section. A final paragraph presents some special theorems on Moutard-Čech hyperquadrics.

J. L. Vanderslice (College Park, Md.).

Calugareanu, Georges, et Gheorghiu, Gh. Th. Sur l'interprétation géométrique des invariants différentiels fondamentaux en géométrie affine et projective des courbes planes. Bull. Math. Soc. Roumaine Sci. 43, 69–83 (1941). [MF 12730]

The authors study in great detail the osculating cubics at a point of a plane curve and arrive at some interesting geometric interpretations of the affine and projective arc and curvature.

A. Schwartz (State College, Pa.).

Gheorghiu, Gh. Th. Sur les transformations asymptotiques. Bull. Sci. Math. (2) 69, 12–20 (1945). [MF 13625]

The problem to be solved is: given a surface S described by a point x, to find a point x' on the Darboux quadric of order c, such that (i) the surface S' described by x' is an asymptotic transform of S and (ii) the Darboux quadric of index c of S' at x' passes through x. The problem is to be discussed in greater detail in a forthcoming memoir. Here certain invariants of the transformation are exhibited and interpreted and special cases of the main problem are completely solved, namely: (i) $c=0$ (Lie quadric), (ii) S and S' projectively applicable, (iii) S and S' in projective correspondence and (iv) S and S' with the same Darboux quadric at corresponding points.

J. L. Vanderslice.

Anglade, E. Points caractéristiques des quadriques de Darboux et propriétés de certains quadrilatères gauches introduits en géométrie différentielle projective des surfaces. Bull. Soc. Math. France 72, 1–26 (1944). [MF 13216]

The first part of the paper presents certain properties of the characteristic points P_i of any Darboux quadric associated with a point O of a surface in 3-space. For example, the planes P_1OP_2, P_3OP_4 cut the tangent plane in conjugate directions. Particular attention is given to the case when the surface is isothermo-asymptotic. There follows a study of pairs of skew quadrilaterals on a quadric. Using two methods of proof proposed by Gambier and Vincensini, respectively, it is shown that the sides cut each other harmonically. Much use is made of this theorem in the rest of the paper. Thus the sides of the quadrilateral of Demoulin divide harmonically the sides of that quadrilateral on the Lie quadric having the directrices of Wilczynski for diagonals. Other applications are to W-congruences and to pairs of linear complexes in involution.

J. L. Vanderslice.

Wilkins, J. Ernest, Jr. A special class of surfaces in projective differential geometry. II. Duke Math. J. 12, 397–408 (1945). [MF 12611]

[Part I appeared in the same J. 10, 667–675 (1943); these Rev. 5, 158]. The paper studies a number of newly defined configurations in connection with the geometry of a surface, in three-dimensional projective space, satisfying the condition $\beta\psi^2-\gamma\varphi^2=0$ mentioned by Lane [Bull. Amer. Math. Soc. 33, 195–200 (1927)]. Typical configurations considered are Grove's conjugal quadrics, Bell's D_k curves and Wu's asymptotic section quadrics and chord section quadrics. Among the results obtained are conditions for the coincidence of some of these configurations.

A. Fialkow.

Rozet, O. Sur les propriétés infinitésimales projectives de certaines variétés V_3 appartenant à un espace linéaire S_4. Bull. Soc. Roy. Sci. Liége 10, 554–559 (1941). [MF 13081]

Rozet, O. Sur les propriétés infinitésimales projectives de certaines variétés à trois dimensions appartenant à un espace à cinq dimensions. Bull. Soc. Roy. Sci. Liége 11, 519–523 (1942). [MF 13120]

Galvani, Octave. Sur la connexion ponctuelle euclidienne des congruences d'éléments linéaires. C. R. Acad. Sci. Paris 218, 264–266 (1944). [MF 13393]

The author has studied the induced Euclidean connections of a family of elements of contact in a Euclidean space of n dimensions [same C. R. 216, 23–25, 519–521 (1943); these Rev. 5, 158]. The case of a congruence of linear elements in ordinary space is now worked out in detail. The curvature (scalar) and the torsion (vector) of the connection are interpreted in terms of differential invariants of the congruence. It is proved, for instance, that the curvature

is equal to the negative of the product of the reciprocals of the focal distances. If the congruences have the same connection of nonzero curvature, their polar surfaces are applicable. *S. Chern* (Princeton, N. J.).

Thomas, T. Y. Absolute scalar invariants and the isometric correspondence of Riemann spaces. Proc. Nat. Acad. Sci. U. S. A. **31**, 306–310 (1945). [MF 13437]

In a Riemann space ("im Kleinen") consider the set of all absolute scalar invariants. If p ($\leqq n$) of these are functionally independent and if the other invariants are functionally dependent on them, the space is said to be of category p. In this paper the following problem is proposed: to determine for pairs of Riemann spaces of the same dimension a set of scalar relationships which imply that the spaces are isometric. In order for there to be isometry, it is necessary that the two spaces be of the same category; the form of these relationships may be expected to depend upon this common category. The problem is solved for n-dimensional spaces of category n, and for two-dimensional spaces of categories 2, 1 and 0. *C. B. Allendoerfer.*

Ruse, H. S. Sets of vectors in a V_4 defined by the Riemann tensor. J. London Math. Soc. **19**, 168–178 (1944). [MF 13639]

The components of the Riemann tensor at a nonsingular point of a Riemannian V_n define a quadratic complex in the space at infinity in the affine tangent space A_n associated with the point. The author continues his study of this configuration for $n=4$, at the same time verifying some previous results of K. W. Lamson. The singular surface of the complex is now a Kummer's quartic surface of 16 nodes and 16 tropes. The 16 nodes determine 16 contravariant vectors in A_n and since the complex cone of a node is doubly degenerate each node defines also a covariant vector, the complex plane of the node. A like treatment of tropes leads to the complex point of a trope. In an Einstein V_4 the Riemann complex is self-polar. The 16 node vectors can be arranged into 8 sets of orthogonal ennuples which correspond to the 8 Rosenhain tetrahedra. The complex planes of the nodes and the complex points of the tropes form a 16_6 configuration. It degenerates when the curvature scalar of the Einstein space vanishes, the complex points of the tropes being the intersection of 4+4 generators of the fundamental quadric. *J. L. Vanderslice.*

Lichnerowicz, André. Sur les espaces riemanniens complètement harmoniques. Bull. Soc. Math. France **72**, 146–168 (1944). [MF 13225]

This is the detailed treatment of results previously announced by the author [C. R. Acad. Sci. Paris **218**, 436–437, 493–495 (1944); these Rev. **6**, 216]. Normal coordinates and the method of normal extension of tensors are employed, by means of which the condition is expressed analytically. Besides the results quoted in previous reviews it is proved that a completely harmonic Riemannian space is an Einstein space and has constant quadratic scalar curvature. If such a space has nonzero curvature, it is indecomposable, etc. *S. Chern* (Princeton, N. J.).

Knebelman, M. S. On the equations of motion in a Riemann space. Bull. Amer. Math. Soc. **51**, 682–685 (1945). [MF 13604]

In a differentiable V_n let there be defined an infinitesimal mapping of V_n into itself by means of the relation $\bar{x}^i=x^i+\xi^i(x)\delta t$. Let $T(x)$ be any tensor in V_n and let $\bar{T}(\bar{x})$ be the transformed tensor. The "variation" of T with respect to the mapping is defined as $\lim_{\delta t\to 0}\{T(\bar{x})-\bar{T}(\bar{x})\}/\delta t$. It is proved that in an affinely connected space variation and covariant differentiation are commutative if and only if the mapping is an affine collineation. Furthermore, such a space admits a group of collineations if and only if it admits a group of mappings for which the variations of the curvature tensor and of its derivatives are zero. If V_n is a Riemann space which admits an infinitesimal mapping, it follows from Killing's equations that variation and covariant differentiation are commutative, and that the variations of the curvature tensor and its derivatives are zero. Conditions are also established which tell whether or not a Riemann space admitting a group of affine collineations also admits a group of motions. *C. B. Allendoerfer.*

Moghe, D. N. On isotropic spaces. J. Univ. Bombay (N.S.) **12**, part 3, 1–3 (1943). [MF 12939]

An isotropic V_n is one in which the Riemannian curvature at any point is the same for all orientations. By Schur's theorem the curvature is constant over the whole space. The author refers to this theorem, yet appears to think that the curvature need not be constant. The paper contains mistakes, misprints and undefined symbols. Much of the analysis seems to be formally correct, but it is difficult to attach any meaning to it. Reference is made to Proc. Indian Acad. Sci., Sect. A. **10**, 275–278 (1939); these Rev. **1**, 125. *H. S. Ruse* (Southampton).

Théodoresco, N. Sur les géodésiques de longueur nulle de certains éléments linéaires finslériens. Bull. École Polytech. Bucarest [Bul. Politehn. Bucureşti] **12**, 9–16 (1941). [MF 13551]

Through a consideration of the characteristics of a linear partial differential equation of order greater than two there can be associated with it a Finsler metric. However, in the resulting geometry the differential equations for the Finsler geodesics (the autoparallels in the now classical Cartan presentation) become infinite on the isotropic cone and cannot be used to define the geodesics of length zero. The author manages to put these equations in a form having sense even on the isotropic cone by modifying the fundamental tensor. *J. L. Vanderslice* (College Park, Md.).

Théodoresco, N. La géométrie de l'équation des ondes. I. Bull. Math. Soc. Roumaine Sci. **41**, 101–109 (1939). [MF 12852]

Given the general wave equation in n dimensions,

$$a^{ij}\frac{\partial^2 u}{\partial x^i\partial x^j}+b^i\frac{\partial u}{\partial x^i}+cu=0,$$

its local characteristic cones are parallel in the Riemann metric of the associate tensor a_{ij} of a^{ij}, and in general the only Riemann metric for which this property holds is of the form λa_{ij}, where λ is a scalar function. In certain cases other metrics exist having the parallelism property but in which the characteristic cones are no longer isotropic. The author is trying to show that the selection of a geometry for the wave equation depends primarily on affine rather than on metric considerations. [Cf. the three following reviews.] *J. L. Vanderslice* (College Park, Md.).

Théodoresco, N. La géométrie de l'équation des ondes. II. Bull. Math. Soc. Roumaine Sci. **42**, no. 1, 79–89 (1940). [MF 12740]

The author discusses further the question of finding all Riemann metrics in which the characteristic cones of the

general wave equation are parallel [see the preceding review]. His procedure simultaneously solves the following problem: given the metric, to find all fields of cones parallel with respect to it. The cases of constant and zero curvature and the statical line element of Einstein get special attention. By examples it is shown how a conformal change of metric may either augment or diminish the number of solutions of the problem. *J. L. Vanderslice.*

Théodoresco, N. La géométrie de l'équation des ondes. III. Bull. Math. Soc. Roumaine Sci. **43**, 59–68 (1941). [MF 12729]

Starting with a general wave equation [see the two preceding reviews] and an arbitrary affine connection Γ^i_{jk}, the former can be written in the tensor form

$$a^{ij}u_{,ij}+a^iu_{,i}+au=0,$$

utilizing covariant derivatives with respect to Γ. By insisting that the characteristic cones be parallel with respect to Γ, the latter specializes to a semi-metric connection depending on the Christoffel symbols of a_{ij}, an arbitrary torsion tensor, and an arbitrary vector λ_i. A demand that the bicharacteristics be self-parallel specializes the form of the torsion tensor. In particular, the conditions are satisfied when the torsion vanishes, the connection becoming a Weyl connection. The arbitrary λ_i can then be selected in a unique way to make the a^i of the wave equation vanish.

J. L. Vanderslice (College Park, Md.).

Théodoresco, N. La géométrie de l'équation des ondes. IV. Bull. Math. Soc. Roumaine Sci. **44**, 71–84 (1942). [MF 12751]

After a partial restatement of results of previous papers [see the three preceding reviews] and the introduction of an affine connection for the purpose of writing the general wave equation in invariant form, the author presents a set of six postulates for defining the covariant differential of any tensor. They are equivalent to but not identical with those of Schouten's "Ricci-Kalkül" and lead to the same formalism, including the possible use of different connections for contravariant and covariant tensor indices. His object is to exhibit the most general formal background for the study of the general wave equation regarded as a geometric object in the sense of Veblen. *J. L. Vanderslice.*

Seetharaman, V. The geometry of partial differential equations of the second order. Proc. Indian Acad. Sci., Sect. A. **21**, 211–217 (1945). [MF 13400]

The author studies the invariant theory of the system

$$\frac{\partial^2 x^i}{\partial u^\alpha \partial u^\beta}+H^i_{\alpha\beta}\left(u, x, \frac{\partial x}{\partial u}\right)=0,$$

adopting the viewpoint of D. D. Kosambi. A covariant differentiation process depending on two connections $\gamma^i_{\alpha j}$ and $\Gamma^\alpha_{\beta\gamma}$ is assumed to exist. The object of the paper is to obtain the tensor invariants and operators of this system by applying the method of elimination to the transformation laws of H, γ and Γ. The end result is a set of eleven basic tensor invariants including those of Kosambi and three which are apparently new. *J. L. Vanderslice.*

Davies, E. T. Motion in a metric space based on the notion of area. Quart. J. Math., Oxford Ser. **16**, 22–30 (1945). [MF 12788]

The author obtains the differential equations and their conditions of integrability for the existence of an infinitesimal motion in a space whose metric depends on the element of area. The Lie derivative that the author uses is, geometrically, the rate of variation used by the reviewer in the corresponding problem for Finsler spaces.

M. S. Knebelman (Pullman, Wash.).

Wagner, V. Homological transformations of Finslerian metric. C. R. (Doklady) Acad. Sci. URSS (N.S.) **46**, 263–265 (1945). [MF 12946]

It is known that Finslerian geometry in an n-dimensional space is the geometry of a field of hypersurfaces (the indicatrices) in the tangent centro-affine Euclidean spaces. A particular class of transformations (called homological transformations by the author) in the tangent spaces is considered, which change the Finslerian metric but not the extremal curves. Problems concerning the effect of this transformation on the properties of the Finslerian metric are studied and invariants are constructed which remain unchanged under the homological transformations.

S. Chern (Princeton, N. J.).

Lee, Hwa-Chung. On even-dimensional skew-metric spaces and their groups of transformations. Amer. J. Math. **67**, 321–328 (1945). [MF 12436]

This is a continuation of the author's paper [same J. **65**, 433–438 (1943); these Rev. **5**, 15] on spaces with a skew fundamental tensor. Here he considers three groups of transformations in connection with such a space: the conformal group whose transformations result in the multiplication of the fundamental tensor by a factor, the special conformal group for which this factor is constant, and the group of automorphisms for which this factor is one. What makes these considerations especially interesting is that it is possible to establish connections with classical concepts. We mention a generalization of the Poisson brackets and of the Jacobi identity and the fact that the automorphisms of a "flat" space correspond in a sense to canonical transformations of $2n$ variables. For the interpretation of the special conformal group the author introduces as "Hamiltonian congruences" of curves solutions of a system of differential equations which in the simplest case reduce to the Hamiltonian equations of analytical mechanics; he then finds that the special conformal transformations are exactly the transformations under which the totality of Hamiltonian congruences is invariant. *G. Y. Rainich.*

Henderson, Archibald. The geometry of tensors of the first order. J. Elisha Mitchell Sci. Soc. **61**, 33–47 (1945). [MF 12985]

The main body of the paper is concerned with the transformation of a vector in 3-dimensional Euclidean space, the components of which are given initially with respect to a set of rectangular Cartesian coordinates. The covariant and contravariant components of the vector are calculated for transformations to oblique Cartesian, to cylindrical polar, to spherical polar and to confocal coordinates, and are exhibited in a set of diagrams drawn for the author by R. M. Trimble. *H. S. Ruse* (Southampton).

Marchetti, Luigi. Sulla costanza dei tensori ε. Pont. Acad. Sci. Acta **4**, 15–20 (1940). (Italian. Latin summary) [MF 12851]

A derivation of the fact that the covariant derivatives of the Kronecker tensors ϵ are zero. *S. Chern.*

NUMERICAL AND GRAPHICAL METHODS

Corrington, Murlan S., and Miehle, William. Tables of Bessel functions $J_n(x)$ for large arguments. J. Math. Phys. Mass. Inst. Tech. 24, 30–50 (1945). [MF 12314]

Tables of $J_n(ms)$ are given for $n=0(1)10$, $s=1(1)20$ and $m=1(1)10$ to five decimal places. A second set of tables includes $n=0(1)10$, $s=1(1)20$ and $m=\pi, 2\pi, \cdots, 5\pi$ also to five decimal places. These computed values have been arranged in tables with increasing s for use in the summation of the Schlömilch series which occur in the theory of frequency modulation. *Author's summary.*

Carsten, H. R. F., and McKerrow, N. W. The tabulation of some Bessel functions $K_\nu(x)$ and $K_\nu'(x)$ of fractional order. Philos. Mag. (7) 35, 812–818 (1944). [MF 12955]

The temperature field within a cylindrical rod subjected to a sudden change of temperature is derived with the aid of the Laplace transformation and is found to approximate, near the center of the rod, to the function

$$u(x,t)=2(\tfrac12\pi)^{-\frac12}\sum_{n=0}^{\infty}\{(2x^2/\alpha)^n/(n!)^2\}(d^n/dt^n)\{T^{\frac12}e^{-T}K_{\frac12}(T)\},$$

where $T=1/\alpha t$ and $\alpha=2(2a/R)^2$. Tables of $K_s(x)$, generally with 5 significant figures, are given for $2s=1(2)5$, $x=.1(.1)5, 6, 8, 10$, and for $4s=-3(2)3$, $x=.1(.1)5, 6, 8, 10$. Values of $K_{\frac14}(x)$ and $K_{\frac34}(x)$ were computed by Lagrange's interpolation formula, using tabulated values of $K_0, K_1, K_2, \cdots$. The values of $K_{n+\frac12}(x)$ were derived from the series

$$K_{n+\frac12}(x)=(\pi/2x)^{\frac12}e^{-x}\sum_{s=0}^{n}\frac{(n+s)!}{s!(n-s)!}(2x)^{-s}$$

and use was made of the relations $K_{-m}(x)=K_m(x)$, $K_{m+1}(x)-K_{m-1}(x)=2(m/x)K_m(x)$ to find $K_m(x)$ for other values of m. *H. Bateman* (Pasadena, Calif.).

Bickley, W. G., and Miller, J. C. P. Notes on the evaluation of zeros and turning values of Bessel functions. I. Introductory. Philos. Mag. (7) 36, 121–124 (1945). [MF 13279]

That zero of the function $\mathfrak{C}_n(x)=J_n(x)\cos\alpha-Y_n(x)\sin\alpha$ for which the first term of the McMahon expansion is $\frac14 a\pi$ is denoted by $c_{n,a}$. The notation $y_{n,s}=c_{n,2n+4s-3}$, $j_{n,s}=c_{n,2n+4s-1}$ gives the zeros of $Y_n(x)$, $J_n(x)$, respectively. The McMahon series is used for the larger zeros but inverse interpolation is used in the range 0–25 for J_n, 0–20 for Y_n. The McMahon series is asymptotic and the accuracy obtainable is limited by the magnitude of the smallest term. It is remarked that the first zero of $J_{20}(x)$ lies outside the tabular range ($j_{20,1}=25.41714\cdots$), while 8 terms of the McMahon series do not yield the required accuracy until x exceeds 100. Additional terms of this series and the related ones for zeros of $J_n'(x)$, $Y_n'(x)$ (denoted here by $j'_{n,s}$, $y'_{n,s}$, respectively) were thus needed, as only 3 are usually given. The use of Debye's series and the equivalent one of Meissel requires a table of arctan x to many decimals and at small intervals. On the recommendation of one of the authors an appropriate table has been made by the New York Mathematical Tables Project. *H. Bateman* (Pasadena, Calif.).

Bickley, W. G., and Miller, J. C. P. Notes on the evaluation of zeros and turning values of Bessel functions. II. The McMahon series. Philos. Mag. (7) 36, 124–131 (1945). [MF 13280]

[Cf. the preceding review.] If $a=4s+2n-1-4\alpha/\pi$ the larger zeros of $\mathfrak{C}_n(x)$ are given by the asymptotic expansion

$$c_{n,a}\sim\beta\{1-\sum_{m=0}^{\infty}(A_{2m+1}/(2m+1)!)(1/8\beta^2)^m\}.$$

Expressions for the coefficients A_{2m+1} are given for $m=0(1)6$. The numbers involved become smaller when use is made of the auxiliary quantity $\lambda=(\mu-1)/24=(4n^2-1)/24$. For numerical work the new expressions may be less convenient than the old ones when n is an integer but when n is half an odd integer they are advantageous. Numerical results are given for $n=2, 20, 1\frac12, 2\frac12$. The last two cases are useful in giving solutions of a transcendental equation of type $\tan(x+\alpha)=x$. Details of the application of the McMahon series are given and tables are included for the coefficients B in a series

$$C_{n,a}\sim\tfrac14\pi a-\frac{B_1}{a}-\frac{B_1B_2}{a\cdot a^2}-\frac{B_1B_2B_3}{a\cdot a^2\cdot a^3}-\cdots,$$

n having values $0(1)20$ and $\frac12(1)10\frac12$; $\frac13(\frac13)\frac23$; $\frac14(\frac12)\frac34$. *H. Bateman* (Pasadena, Calif.).

Bickley, W. G. Notes on the evaluation of zeros and turning values of Bessel functions. III. Interpolation by Taylor series. Philos. Mag. (7) 36, 131–133 (1945). [MF 13281]

[Cf. the preceding review.] In tabulating $J_m(x)$ and $Y_m(x)$ at intervals of 0.1 in x, starting from Meissel's values at unit intervals, Comrie's method is used, in which the derivatives are calculated by a central difference formula of type

$$2^nJ_m^{(n)}(x)=J_{m-n}(x)-nJ_{m-n+2}(x)+\cdots.$$

Having found a rough approximation x to a zero of $J_m(x)$ which makes $J_m(x)$ small, interpolation can be used as the coefficients in the Taylor series can be derived. An example is given with $x=30$ and $m=20$ when $J_{20}(30)=+0.00483\cdots$. Thirteen decimals are kept in the calculations and twelve are given for the zero. Checks are given. *H. Bateman.*

Miller, J. C. P., and Jones, C. W. Notes on the evaluation of zeros and turning values of Bessel functions. IV. A new expansion. Philos. Mag. (7) 36, 200–206 (1 plate) (1945). [MF 13343]

[Cf. the preceding review.] If $xp=J_n(x)/J_{n-1}(x)$, $xq=J_n(x)/J_{n-1}(x)$, $h=x_0-x$, $r=h/x$, where x is an approximation to a zero of $J_n(x)$ and x_0, different from zero, is the exact value of the zero desired, the method to be discussed consists in developing r as a power series in either p or q with coefficients that are functions of x. Thus r involves x both explicitly and also implicitly through p or q. Provided that x_0 does not occur explicitly the method is one of direct calculation.

When $r=-a_1p-a_2p^2-a_3p^3-\cdots$, with $a_1=1$, $2a_2=(2n-1)a_1$, a recurrence relation

$$(m+1)a_{m+1}=(2mn-1)a_m-(m-1)x^2a_{m-1}-xda_m/dx$$

is obtained and the coefficients a_m calculated step by step. Thus a_m is a polynomial in x^2 whose coefficients $C_{m,k}$ are connected by a recurrence relation which is given. There is a corresponding analysis when r is expanded in powers of q. In this way the McMahon series are found for $n=\frac13, \frac23$. Tables of a_2, a_3, a_4, a_5, a_6 are given for $n=-10(1)10$ and for $n=-\frac{19}{2}(1)\frac{19}{2}$; $n=-\frac23(\frac43)\frac23$; $n=-\frac34(\frac12)\frac34$. The initial approach to a zero is discussed and an example is given of the use of the formulae. *H. Bateman* (Pasadena, Calif.).

Bickley, W. G., and Miller, J. C. P. Notes on the evaluation of zeros and turning values of Bessel functions. V. Checks. Philos. Mag. (7) 36, 206–210 (1945). [MF 13344]

[Cf. the preceding review.] In table making, the checking of the computations is very important and may be more troublesome than the original calculations. Methods of checking the computed values of $j_{n,s}$ and $y_{n,s}$ are considered. For a fixed n and for large s differencing may be used. Since the McMahon expansions have common coefficients it is possible to combine the zeros of Y_n and J_n into a single series, as is exemplified in table I with the zeros of Y_0, J_0; Y_1, J_1; Y_2, J_2 and Y_3, J_3, five decimals being given. The combination has an effect similar to that of halving the interval. *H. Bateman* (Pasadena, Calif.).

Bickley, W. G. The tabulation of Mathieu functions. Mathematical Tables and other Aids to Computation 1, 409–419 (1945). [MF 12681]

The author gives a concise summary of the published numerical results on the various types of solutions of the Mathieu equation $y''+(a-2\theta\cos 2x)y=0$. Characteristic numbers (values of a for which periodic solutions exist) and also some of the coefficients of the associated Fourier series have been computed by several authors. For nonperiodic solutions, in which a and θ are both preassigned, only one reference is available. This includes a table of the characteristic exponent μ, where the solution is expressed in the form $y=Ae^{\mu x}\Phi(x)+Be^{-\mu x}\Phi(-x)$, Φ being periodic in x. A useful list of discrepancies in the existing tables completes the paper. *M. C. Gray* (New York, N. Y.).

Neushuler, L. J. On tabulation of systems of non-explicit functions of two variables. C. R. (Doklady) Acad. Sci. URSS (N.S.) 44, 360–364 (1944). [MF 12568]

The author considers the problem of tabulating non-explicit functions of two variables x_i and x_j defined by the equations

$$F_i(u, v, x_i)=0, \quad F_j(u, v, x_j)=0,$$

with a view to condensation of the tabular material and the inclusion of various first differences for interpolation. Seven different arrangements are suggested but none is illustrated. *D. H. Lehmer* (Aberdeen Proving Ground, Md.).

Täht, A. Über die Lösung der Keplerschen Gleichung. Bull. Inst. Astr. Acad. Sci. URSS no. 53, 478–480 (1944). (Russian. German summary) [MF 12889]

The author explains and illustrates an iterative process for solving Kepler's equation

$$x-a\sin x+b=0,$$

based on the four term truncated expansion by Taylor's series of the left member about an approximate root. Solving the resulting cubic equation leads to a correction for the root and the process is repeated. Each error is approximately the 4th power of the error at the preceding stage. *D. H. Lehmer* (Aberdeen Proving Ground, Md.).

Fry, Thornton C. Some numerical methods for locating roots of polynomials. Quart. Appl. Math. 3, 89–105 (1945). [MF 12647]

First part: detailed discussion of methods based on matrix algebra. The discussion is centered around two papers by Duncan and Collar [Philos. Mag. (7) 17, 865–909 (1934); 19, 197–219 (1935)]. The relation of these papers to Bernoulli's and Graeffe's methods of solution is brought out. The author, as mathematician for the Bell Telephone Laboratories, is able to give a reasoned evaluation of the respective merits of the various methods in actual practice. The mathematical staff of the laboratory has contributed variants of its own, which are outlined.

Second part: discussion of methods based on conformal mapping which all go back in some form or other to Cauchy's well-known contour theorem. These methods have been utilized by the Bell Laboratories in the construction of an elaborate machine, the isograph, which is described and which gives the mechanical solution of numerical equations of degree less than eleven to about 1% accuracy. Both real and complex roots are found, but coefficients are restricted to be real.

These two parts were written some years ago, and are brought up to date by a postscript by R. L. Dietzold. One significant advance has been the construction of "relay computers" (not described) which "enable the algebraic operations to be performed on complex numbers with the ease that the same operations are performed on real numbers with a mechanical computing machine." This apparently makes it possible to solve "mechanically" equations with complex coefficients without the passage through equations with real coefficients. *A. J. Kempner.*

James, Glenn. Evaluation of real roots by means of lower degree equations. Nat. Math. Mag. 19, 375–384 (1945). [MF 13286]

The theory explained leads to the following rule for the approximation to real roots of an algebraic equation. (I) Given $f(x)=a_0x^n+a_1x^{n-1}+\cdots+a_n=0$, consider either one of the following equations

$$\text{(A)}\qquad f(h_1)+\frac{x-h_1}{n}f'(h_1)=0,$$

$$\text{(B)}\qquad f(h_1)+2\frac{x-h_1}{n}f'(h)+\frac{(x-h_1)^2}{n(n-1)}f''(h_1)=0.$$

Choose for h_1 an approximation to a desired real root of $f(x)=0$ and, with this value h_1, solve either (A) or (B) for x. Let R_1 be the root of whichever equation was chosen. In case (A), let $h_2=(R_1+(n-1)h_1)/n$; in case (B), let $h_2=(2R_1+(n-2)h_1)/n$. (II) Repeat (I) with h_2 instead of h_1. (III) Repeat the process i times, where i is taken large enough to make $\frac{1}{2}|h_{i+1}-R_i|$ less than the numerical error permitted in the root. (IV) Then $x=\frac{1}{2}(h_{i+1}+R_i)$ will be a root of $f(x)=0$ within the error permitted. Case (A) involves only rational operations.

The paper contains an indication of the application of the method to the more difficult problem of finding complex roots, and is also applied to transcendental equations, such as $x^2-\cos x=0$. *A. J. Kempner* (Boulder, Colo.).

Babini, J. On the transformation of the method of Graeffe. Publ. Inst. Mat. Univ. Nac. Litoral 5, 45–49 (1945). (Spanish) [MF 13721]

In Graeffe's method for solving the polynomial equation $f(x)=0$, the polynomial is replaced by $F(X)=f(x)f(-x)$, $X=-x^2$, and the process is iterated. The author proposes to put $X=x^3$, $F(x)=f(x)f(\omega x)f(\omega^2x)$, where ω is a primitive cube root of unity, and iterate this process. He suggests that this may be advantageous for equations of degree four or less, and gives numerical examples. *R. P. Boas, Jr.*

Lehmer, D. H. The Graeffe process as applied to power series. Mathematical Tables and other Aids to Computation 1, 377–383 (1945). [MF 12440]

Graffe's method, as modified by Brodetsky and Smeal [Proc. Cambridge Philos. Soc. 22, 83–87 (1924)], is extended to the problem of finding a pair of complex zeros of an entire function $f(z)=1+\sum_{n=1}^{\infty}a_n z^n$, a_n real. Two sequences $\{A_r^{(m)}\}$, $\{B_r^{(m)}\}$ are defined by the relations

$$A_r^{(0)}=a_r, \quad B_r^{(0)}=(r-1)a_{r-1},$$

$$A_r^{(1)}=(-1)^r a_r^2+2\sum_{\nu=0}^{r-1}(-1)^\nu a_\nu a_{2r-\nu},$$

$$B_r^{(1)}=\sum_{\nu=0}^{r-1}(-1)^\nu(2r-2\nu-1)a_\nu a_{2r-\nu-1},$$

and for $m>1$

$$A_r^{(m)}=(-1)^r(A_r^{(m-1)})^2+2\sum_{\nu=0}^{r-1}(-1)^\nu A_\nu^{(m-1)}A_{2r-\nu}^{(m-1)},$$

$$B_r^{(m)}=\sum_{\nu=0}^{2r}(-1)^\nu A_\nu^{(m-1)}B_{2r-\nu}^{(m-1)}.$$

Let α_k denote the reciprocals of the zeros of $f(z)$, arranged in nonincreasing order of their moduli. If $|\alpha_1|>|\alpha_2|>\cdots$, then, approximately, $\alpha_k=A_k^{(m)}/B_k^{(m)}$ when m is sufficiently large. If the zeros appear in conjugate pairs, $\alpha_{2k-1}=\rho_k e^{i\theta_k}$, $\alpha_{2k}=\rho_k e^{-i\theta_k}$, $\cdots$, then, approximately, with $s=2^m$ and m sufficiently large,

$$\rho_1=|A_2^{(m)}|^{1/2s}, \quad \rho_2=|A_4^{(m)}/A_2^{(m)}|^{1/2s}, \cdots;$$
$$2\cos\theta_1=\rho_1 B_2^{(m)}/A_2^{(m)},$$
$$2\cos\theta_2=\rho_2(B_4^{(m)}/A_4^{(m)}-B_2^{(m)}/A_2^{(m)}), \cdots.$$

Other cases, as well as the practical difficulties of applying the method to entire functions, are discussed. The paper closes with an application to the calculation of two pairs of conjugate complex roots for the equation $J_1^2(z)=J_0(z)J_2(z)$, where $J_n(z)$ is the Bessel function of the first kind.

M. Marden (Milwaukee, Wis.).

Duncan, W. J. Some devices for the solution of large sets of simultaneous linear equations. Philos. Mag. (7) 35, 660–670 (1944). [MF 11956]

This paper is concerned with a concise arrangement of the work involved in solving a large number $n+r$ of simultaneous linear equations by the "method of submatrices." The principle is the familiar one: solve n (judiciously chosen) equations in terms of the remaining r, substitute these solutions and solve the rest. Examples are given illustrating the technique employed. *A. L. Foster.*

Morris, J., and Head, J. W. The "escalator" process for the solution of Lagrangian frequency equations. Philos. Mag. (7) 35, 735–759 (1944). [MF 12329]

In a previous paper [Aircraft Engrg. 14, 312–314, 316 (1942); these Rev. 4, 148] the authors have described a new method for finding characteristic roots and characteristic vectors satisfying matric equations of the form $AX=\lambda BX$, where A and B are symmetric square matrices [the condition of symmetry should be added in the review quoted above]. The present paper contains a more elaborate exposition of the method, together with several improvements, checks, etc. *W. Feller* (Ithaca, N. Y.).

Waugh, F. V. A note concerning Hotelling's method of inverting a partitioned matrix. Ann. Math. Statistics 16, 216–217 (1945). [MF 12914]

[For Hotelling's method cf. the same Ann. 14, 1–34, 440–441 (1943); these Rev. 4, 202; 5, 245.] Let a, b, c, d be square matrices of order p and write the inverse m^{-1} of the partitioned matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

in the form

$$\begin{pmatrix} A & C \\ B & D \end{pmatrix}.$$

The author observes that m^{-1} can be computed by the following formulas involving only inversions of two matrices of order p:

$$A=a^{-1}-a^{-1}bB, \quad B=-Dca^{-1},$$
$$C=-a^{-1}bD, \quad D=(d-ca^{-1})^{-1}.$$

W. Feller (Ithaca, N. Y.).

Greville, Thomas N. E. The general theory of osculatory interpolation. Trans. Actuar. Soc. Amer. 45, 202–265 (1944). [MF 11971]

The contents are best described by the following extracts from the author's summary. The existing literature on osculatory interpolation is almost entirely devoted to the consideration of individual formulas and particular families of formulas. Nowhere has the general underlying theory been completely and systematically developed. In this paper an expression is developed for the most general osculatory interpolation formula with any desired order of contact. The general expression is then restricted by imposing reasonable basic conditions, and the particular cases of first and second order contact are considered in detail. Criteria are offered to furnish guidance in selecting from the great variety of possible formulas one which is appropriate for use in a given situation. Special attention is given to the possibility of combining interpolation and smoothing in a single operation. A numerical example is worked out to illustrate the various principles stated, and, finally, consideration is given to the procedure to be adopted for interpolating near the ends of the range of "given" values where the usual symmetric formulas are not completely determinate. Only interpolation with equal intervals is considered. *W. E. Milne* (Corvallis, Ore.).

Greville, Thomas N. E. The general theory of osculatory interpolation. Trans. Actuar. Soc. America 46, 83–101 (1945). [MF 13541]

This is the written discussion, customary in the Actuarial Society, of the paper reviewed above.

Beers, Henry S. Six-term formulas for routine actuarial interpolation. Record Amer. Inst. Actuar. 33, 245–260 (1944). [MF 12424]

The author considers interpolation, by means of formulas expressing the interpolated value as a weighted average of six given ordinates, upon a function for which ordinates are given at equal intervals, for the purpose of subdividing these intervals into fifths. He rejects the traditional osculatory formulas on the ground that it is only the values of the function at certain discrete points which are of any concern in practical actuarial work, and that the characteristics of a hypothetical smooth curve fitted to the given data have little relevance. Instead, he adopts a criterion of smoothness based on minimizing the sum of the squares of the fifth

differences of the interpolated series. Requiring that the formula be correct to fourth differences, he derives the six-term formula producing the smoothest interpolated series on the basis of this criterion, on the assumption that the fifth differences of the given ordinates can be considered as independent random variables with zero mean and equal variance. The properties of this formula are compared with those of other six-term formulas, and tables of coefficients are given for computing the interpolated values from the given ordinates. These include special coefficients for use in the two interpolation intervals at each end of the series in which the symmetrical formula used elsewhere does not apply. *T. N. E. Greville* (Washington, D. C.).

Taylor, William J. Method of Lagrangian curvilinear interpolation. J. Research Nat. Bur. Standards 35, 151–155 (1945). [MF 13510]

A compact expression is given for the Lagrangian interpolation coefficient in the case of equal intervals.
T. N. E. Greville (Washington, D. C.).

Sen, D. K. Interpolation and summation formulas and the properties of factorials. I. J. Univ. Bombay (N.S.) 11, part 3, 22–36 (1942). [MF 12184]

The error in any polynomial interpolation formula of degree n which fits the given function at $n+1$ equally spaced points is of the form

$$R=(x-x_0)(x-x_1)\cdots(x-x_n)f^{(n+1)}(\xi)/(n+1)!$$

and is independent of the type of interpolation formula. The error does, however, depend on the interval in which the interpolation is made.

The present paper, the first part of a more extensive investigation of the error, studies the size and location of the successive maxima and minima of the factorial function $x(x-1)(x-2)\cdots(x-n)$. Tables are given for the extreme ordinates up to $n=10$; the asymptotic behavior as $n\rightarrow\infty$ is examined; and the ratio of the largest extreme ordinate to the least extreme ordinate is studied. *W. E. Milne.*

Salzer, Herbert E. Note on interpolation for a function of several variables. Bull. Amer. Math. Soc. 51, 279–280 (1945). [MF 12268]

The formula obtained is a polynomial of total degree n in $p, q, \cdots, r$ such that

$$f(x+ph_1, y+qh_2, \cdots, z+rh_u)=\sum_{s+t+\cdots+u=0}^{n} C_{s,t,\cdots,u}f_{s,t,\cdots,u},$$

where

$$C_{s,t,\cdots,u}=\binom{n-p-q-\cdots-r}{n-s-t-\cdots-u}\binom{p}{s}\binom{q}{t}\cdots\binom{r}{u}.$$

For two dimensions the points used in the formula form a right triangle, in three dimensions a tetrahedron, etc.
W. E. Milne (Corvallis, Ore.).

Salzer, Herbert E. Inverse interpolation for eight-, nine-, ten-, and eleven-point direct interpolation. J. Math. Phys. Mass. Inst. Tech. 24, 106–108 (1945). [MF 13351]

This paper extends the formulas presented in an earlier article [Bull. Amer. Math. Soc. 50, 513–516 (1944); these Rev. 6, 53] to cover the cases listed in the title.
W. E. Milne (Corvallis, Ore.).

Collatz, L., und Zurmühl, R. Glätten und Vertafeln empirischer Funktionen mittels Differenzen. Z. Verein. Deutsch. Ingenieure 88, 511–515 (1944). [MF 12341]

The familiar rough method of fitting a graph to a set of points representing experimental data is to sketch a curve which satisfies the eye as being a sufficiently good fit. The authors' modification consists in: first, plotting against the given x the first and second differences of the given y; second, sketching by eye smooth curves C_1 and C_2 passing approximately through these points; third, replacing the actual first and second differences by the smoother differences read off from C_1 and C_2 and thus replacing the given (x, y) by a new set. Additional points (x, y) may be inserted between these new (x, y) by use of interpolation polynomials. Thus the original set of points (x, y) may be replaced by a denser set through which a smooth graph may be readily drawn. *M. Marden* (Milwaukee, Wis.).

Porta, Livio Dante. On a graphical method of approximation of functions. Math. Notae 4, 227–238 (1 plate) (1944). (Spanish) [MF 12207]

A function $V(x)$ is to be approximated by a function $F(x;\lambda,\mu,\cdots)$ involving the unspecified parameters $\lambda,\mu,\cdots$; the problem is to determine the values of the parameters which provide the best approximation, defined as that which minimizes $\int_a^b|V-F|dx$. This is done by using graphical methods to find the values of the parameters for which the variation of this integral is zero. The procedure is illustrated for certain simple cases. *T. N. E. Greville.*

Lowan, Arnold N., and Salzer, Herbert E. Table of coefficients for numerical integration without differences. J. Math. Phys. Mass. Inst. Tech. 24, 1–21 (1945). [MF 12312]

This table gives to 10 decimal places the values of the integrals of the Lagrangian interpolation coefficients. For the cases of 3-, 4- and 5-point coefficients the tabular interval is 0.01 of the interpolation interval, while for 6- and 7-point coefficients it is 0.1 of the interpolation interval. In all cases, the tabular range covers the entire range of interpolation. *W. E. Milne* (Corvallis, Ore.).

Southwell, R. V. On relaxation methods: A mathematics for engineering science. Proc. Roy. Soc. London. Ser. A. 184, 253–288 (1945). [MF 13356]

Bakerian lecture before the Royal Society.

Amstutz, E. Neue Methoden der analytischen Statik linearer Probleme. Schweiz. Arch. Angew. Wiss. Tech. 9, 101–109 (1943). [MF 13000]

The author first derives formulas for the calculation of the moment, the elastic load, the slope and the integral of moment products for the case of linear beams deflected under load. The method is to divide the length into suitable subintervals in each of which the loading function is assumed to be linear.

The second part of the paper presents two step-by-step methods for solving the differential equation

$$M''\pm M/r^2=-p,$$

where p is assumed linear in each subinterval. The third part gives a method for solving problems of stability associated with the differential equation when $p=0$, where a solution is possible only for definite values of r.
W. E. Milne (Corvallis, Ore.).

Bitterlich-Willmann, J. Zum Verfahren der schrittweisen Näherung. Z. Angew. Math. Mech. 21, 124–125 (1941). [MF 12153]

Es wird im folgenden eine Lösung der Differentialgleichung $y''+F(x)y=0$ unter gewissen Voraussetzungen über $F(x)$

nach der Methode der schrittweisen Näherung berechnet; für die so erhaltene Reihe werden bestimmte Eigenschaften ihrer Glieder festgestellt, und es wird eine Restabschätzung gegeben. *Author's summary.*

Schweikert, G. Zur Theorie und Konstruktion der Geschossflugbahn. Z. Angew. Math. Mech. **24**, 49–63 (1944). [MF 13190]

This paper begins with the usual equations for the path of a projectile,

$$\dot{v} = -W_T - g \sin \theta, \quad v^2/\rho = v\dot{\theta} = W_N - g \cos \theta,$$

where W_T and W_N are the tangential and normal components of the air resistance and θ is the angle between the horizontal (x-axis) and the path. It shows how to approximate the path graphically by a broken line formed by arcs of the osculating circles.

If the substitutions $z = \sin \theta$ and $u = \log (v/v_0)$ are made and the variables θ, v and t eliminated, the equation

$$(1) \qquad -(gz + W_T)dz/du + g(1-z^2) = W_N(1-z^2)^{\frac{1}{2}},$$

which is in hodographic form, results. The same substitutions lead from the equations $dy/dt = v \sin \theta$ and $dx/dt = v \cos \theta$ to the equations

$$(2) \qquad (gz + W_T)dy/du = -v_0^2 z e^{2u},$$

$$(3) \qquad (gz + W_T)dx/du = -v_0^2(1-z^2)^{\frac{1}{2}}e^{2u}.$$

If g, W_T and W_N are independent of y, (1) may be solved by quadrature for z in terms of u and then (2) and (3) for y and x in terms of u. Otherwise, z may first be eliminated between (1) and (2) and then y found in terms of u; subsequently, z and x may be found also in terms of u.

Equation (1) reduces to the usual hodographic equation of ballistics on assuming that $W_N = 0$, $W_T = c\delta(y)v^2K(v, y)$, where c is the ballistic coefficient and $\delta(y)$ is the air density. From (2) also follows, on setting $p = -gz - W_T$ and $Y = \frac{1}{2}(v_0^2 - v^2)/g$, the ordinate for flight in a vacuum, that in general the ordinate is

$$y = Y - \int_{v_0}^{v} (vW_T/gp)dv.$$

The paper includes a schematic chart for the mechanical integration of the differential equations and a discussion of the use of (2) and (1) for determining W_T and W_N, and thus the law of resistance, when the path of the projectile is known. *M. Marden* (Milwaukee, Wis.).

Lattin, William J. Note on the Fourier series for several pulse forms. Proc. I. R. E. **33**, 783–784 (1945).

Suppose that a family of periodic functions of period T, depending on a parameter t_1, is such that the Fourier cosine series of a member of the family can be written in the form

$$g(t_1/T)\sum_{k=0}^{\infty} h(kt_1/T) \cos (k\pi t/T),$$

where g and h are known functions. Then the successive coefficients (for various values of t_1/T) can be conveniently read off from a graph of $h(K)$, where $K = kt_1/T$. Functions representing rectangular, triangular and trapezoidal pulses fulfill the conditions; sample graphs are presented. *R. P. Boas, Jr.* (Providence, R. I.).

Popoff, A. A. A new method of integration by means of orthogonality foci. Quart. Appl. Math. **3**, 166–174 (1945). [MF 12653]

The author gives a quick construction for finding approximately the value of

$$\int_0^l \varphi_i(x)\varphi_k(x)dx,$$

where at least φ_i is given graphically. Most of the effort is expended in finding a "scale of φ_k"; hence the method is convenient where the same function recurs in different problems, as in harmonic analysis. *R. L. Dietzold.*

Tea, Peter L. A graphical method for the numerical solution of Fredholm's integral equation of the second kind. J. Math. Phys. Mass. Inst. Tech. **24**, 109–125 (1945). [MF 13352]

The author gives a graphical method for expressing the value of the integral $I = \int_a^b \varphi(y)K(y)dy$ in its equivalent Stieltjes integral form $\int_a^b \varphi(y)dY(y)$, where $Y(y) = \int_a^y K(y)dy$. If $\varphi(y)$ is plotted as ordinate against $Y(y)$ as abscissa, I can be found either graphically or by a planimeter as the area under the curve so defined. The author applies this idea to harmonic analysis and to the graphical solution of Fredholm's equation of the second kind. *B. Friedman.*

Fehlberg, Erwin. Eine Bemerkung zur numerischen Differentiation durch Approximation, ausgeführt am Beispiel der Kugelfunktionen als Approximationsfunktionen. Z. Angew. Math. Mech. **24**, 71–76 (1944). [MF 13192]

To evaluate $\int_{-1}^{1} f(x)P_n(x)dx$, the interval is subdivided and $f(x)$ is replaced in each resulting interval by a second degree parabola. The resulting integral can be evaluated exactly. Auxiliary tables corresponding to a division into five intervals are given for $n \leq 12$. *W. Feller.*

Parodi, Maurice. Deux méthodes de calcul de la fonction $y(t) = \int_0^t f(\tau)g(t-\tau)d\tau$. Revue Sci. (Rev. Rose Illus.) **81**, 171–172 (1943). [MF 13806]

Schneider, Stanislas. Deux autres méthodes de calcul approché de la fonction $y(t) = \int_0^t f(\tau) \cdot g(t-\tau) \cdot d\tau$. Revue Sci. (Rev. Rose Illus.) **82**, 38 (1944). [MF 13801]

The first note suggests a graphical and a numerical method. The second points out that trapezoidal and polynomial approximations can be applied.

Pry, G., et Prigogine, I. Sur le calcul des niveaux énergétiques à l'aide de la méthode de Wentzel-Kramers-Brillouin. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **28**, 652–659 (1942). [MF 13673]

The authors consider the solution of the following equation for E:

$$\int_{-x_1}^{x_1} (E - E_{\text{Pot}}(x))^{\frac{1}{2}}dx = \text{constant},$$

when $E_{\text{Pot}}(x)$ can be expanded in a power series in x. The point x_1 is given by $E - E_{\text{Pot}}(x_1) = 0$. The method consists in introducing the parameter $\beta = x_1^2$ and the variable $u = x/x_1$, so that both E and $E_{\text{Pot}}(x)$ can be expanded in power series in β. The integral is expanded in a power series in β, the coefficients up to the fifth being given in the paper explicitly. The resultant equation must then be solved numerically for β and introduced into the expansion for E to obtain E. *H. Feshbach* (Cambridge, Mass.).

Raulier, Suzanne. Sur le calcul des niveaux énergétiques à l'aide de la méthode de Wentzel-Kramers-Brillouin. Acad. Roy. Belgique. Bull. Cl. Sci. (5) 29, 688–694 (1943). [MF 13864]

This is a sequel to the paper reviewed above, where the first five coefficients of the power series for the integral were evaluated. The author extends the evaluation to the sixteenth term. *H. Feshbach* (Cambridge, Mass.).

Galletti, R. A note on graphs. J. Madras Univ. Sect. B. 15, 19–29 (1943). [MF 12318]

The subtitle of the paper reads: The presentation and analysis of changes in proportion, rates of change and changes in rates of change.

Ahrens, Christian. Netztafel-Nomogramme aus orthogonalen Kreisscharen. Z. Angew. Math. Mech. 24, 87–89 (1944). [MF 13195]

The author discusses nomographic charts formed by two families of mutually orthogonal circles and a third family of circles or straight lines. *E. Lukacs* (Cincinnati, Ohio).

Criner, H. E., McCann, G. D., and Warren, C. E. A new device for the solution of transient-vibration problems by the method of electrical-mechanical analogy. J. Appl. Mech. 12, A-135–A-141 (1945). [MF 13214]

Pérès, Joseph, et Malavard, Lucien. Application du calcul expérimental rhéoélectrique à la solution de quelques problèmes d'élasticité. J. Math. Pures Appl. (9) 20, 363–426 (1941). [MF 13366]

Malavard, Lucien. Sur la solution rhéoélectrique de questions de représentation conforme et application à la théorie des profils d'ailes. C. R. Acad. Sci. Paris 218, 106–108 (1944). [MF 13458]

Hall, D. M., Welker, E. L., and Crawford, Isabelle. Factor analysis calculations by tabulating machines. Psychometrika 10, 93–125 (1 plate) (1945). [MF 12642]

Niethammer, Th. Die Genauigkeit der verschiedenen Zeitbestimmungsmethoden. Verh. Naturforsch. Ges. Basel 51, 29–39 (1940). [MF 13702]

RELATIVITY

***Einstein, Albert. The Meaning of Relativity.** Princeton University Press, Princeton, N. J., 1945. 135 pp. $2.00.

This book is, in the main, a reprint of an earlier work written in 1921, to which an appendix on cosmological theory has been added. Historically, it has great value as an exposition of the arguments used by their founder in establishing the theories of special and general relativity. It is, however, questionable whether more consideration should not have been given in this reissue to the developments of the last 24 years. For example, the special theory and the Lorentz transformation are presented entirely on the basis of a coordinate-system set up by means of ideally rigid measuring rods. This device has been shown to be irrelevant (for example, by the work of E. A. Milne and others) and the special theory would hardly be applicable to the real world if its validity did indeed depend on the concept of a rigid measuring rod. In practice, no such instrument exists, yet the Lorentz transformation between coordinates still holds good.

Rereading Einstein's proof of the identity of mass and energy gives rise to some reflections on the meaning of the term "proof" in mathematical physics. He shows that, from the force acting on unit volume of electricity in a continuous distribution, there can be derived a 4-vector $V_1=(iE, I_x, I_y, I_z)$ representing energy and momentum. Again, in the 4-dimensional space-time of special relativity, a particle can be described by means of a 4-vector $V_2=(imdt/ds,\ mdx/ds,\ mdy/ds,\ mdz/ds)$. The crucial step is then the equating of V_1 to V_2. This has no mathematical basis, but rests on the intuitive notion that V_1 and V_2 must be alternative representations of the same physical entity. It is curious that this result has gained general acceptance when similar, though more recondite, identifications carried out by Eddington in Relativity Theory of Protons and Electrons [Cambridge University Press, 1936] have not yet secured universal assent.

The two chapters on the general theory have rarely been surpassed as succinct accounts of the fundamentals of the theory. But in the new appendix on cosmological theory Einstein gives way to the now fashionable trend which makes cosmology an a priori theory and impels investigators to neglect, if necessary, any awkward observations. The a priori principle here applied is "logical economy" and the neglected observations are Hubble's results on the distribution in depth of the extragalactic nebulae. The object is to dispense with the cosmological constant in the gravitational equations. This omission restricts the range of model universes to those in which (i) the time-scale is very short (10^9 years), (ii) the expansion of the system of extragalactic nebulae begins from an explosion at a point and (iii) the rate of expansion is continually retarded. Einstein himself seems dissatisfied with the first two results; he does not discuss the third at any length. The obvious way of escape is to retain the cosmological constant in the gravitational equations, which would have the added advantage of reconciling the theory with Hubble's observations.

G. C. McVittie (London).

Einstein, Albert, and Straus, Ernst G. The influence of the expansion of space on the gravitation fields surrounding the individual stars. Rev. Modern Phys. 17, 120–124 (1945). [MF 13686]

It is assumed that the gravitational potentials and their first derivatives will be continuous at the boundary of the spherically symmetric star. An approximate solution of the field equations for empty space is obtained and the gravitational potentials thus determined are required to piece together continuously with the known gravitational potentials for a pressure free, spatially constant density of matter. Under these conditions it is possible to show that the Schwarzschild field can be transformed into a solution of the problem. The static nature of the Schwarzschild solution implies that the expansion of space does not make the field surrounding the star time-dependent. [Several misprints and omissions occur in the paper, the most serious being that τ should be given by $\tau=b_0+b_1r^{-1}+b_2r$. This requires several expressions to be modified.] *M. Wyman.*

Drumaux, P. Sur la relation universelle entre la distance et la masse. Ann. Soc. Sci. Bruxelles. Sér. I. 60, 73–79 (1940). [MF 13776]

By means of the field equation the relationship between mass and length is discussed. It is shown that any length l can be considered equivalent to a mass m, where m is given by the relation $m=\frac{1}{2}c^2l/K$. *M. Wyman.*

Drumaux, P. Sur la signification mathématique et physique de la constante cosmologique Λ. Ann. Soc. Sci. Bruxelles. Sér. I. 60, 80–82 (1940). [MF 13777]

The role of the cosmological constant in relativity is discussed and the author states that it cannot be considered as a universal constant of nature. From the way it enters the field equations, it is considered to be a constant of integration. For this reason the constant may have different values in different physical problems. *M. Wyman.*

de Losada y Puga, Cristóbal. Reflections on the theory of relativity. Publ. Inst. Mat. Univ. Nac. Litoral 5, 165–170 (1945). (Spanish) [MF 13727]

An expository article.

Hill, E. L. On accelerated coordinate systems in classical and relativistic mechanics. Phys. Rev. (2) 67, 358–363 (1945). [MF 12660]

Following the lines of investigation initiated by Page [Phys. Rev. (2) 49, 254–268 (1936)], Bourgin [Phys. Rev. (2) 50, 864–868 (1936)] and others, the author discusses the group of transformations to accelerated axes. Starting with the problem of one-dimensional motion, he studies the point transformations which leave the equation $b=0$ (in the classical case) or $b+3va^2/(c^2-v^2)=0$ (in the relativistic case) invariant, where $v=dx/dt$, $a=d^2x/dt^2$, $b=d^3x/dt^3$. In the relativistic three-dimensional case the required group is that of conformal transformations of Minkowski space. The operators generating this group are listed and discussed and their commutators are tabulated. The transition to the classical theory is effected by the limiting process $c\rightarrow\infty$. *A. Schild* (Toronto, Ont.).

de Mira Fernandes, A. Connessioni finite. Portugaliae Math. 4, 203–210 (1945). [MF 13332]

Description of work of A. Einstein and V. Bargmann on bivector fields [Ann. of Math. (2) 45, 1–14, 15–23 (1944); these Rev. 5, 218]. *A. Schwartz* (State College, Pa.).

Schrödinger, Erwin. On distant affine connection. Proc. Roy. Irish Acad. Sect. A. 50, 143–154 (1945). [MF 12684]

The author points out that the "reciprocal" case of the distant affine connection studied by A. Einstein and V. Bargmann [Ann. of Math. (2) 45, 1–14, 15–23 (1944); these Rev. 5, 218] is a natural generalization of some of Einstein's earlier work on "distant parallelism." Necessary and sufficient conditions that the covariant associate of the "distant-affinity tensor" be symmetric or skew-symmetric in an arbitrary frame are deduced and it is pointed out that the conditions for the two cases are not mutually exclusive. The paper concludes with some remarks about the modifications forced on an ordinary infinitesimal affine connection by the assumption of the symmetry conditions and about a field equation proposed by Einstein and Bargmann. *A. Schwartz* (State College, Pa.).

Newing, R. A. Kinematic relativity. Philos. Mag. (7) 36, 113–115 (1945). [MF 13277]

Milne, E. A. Kinematic relativity: A reply to Prof. W. Wilson. Philos. Mag. (7) 36, 134–143 (1945). [MF 13282]

Both authors reply to W. Wilson's criticism [Philos. Mag. (7) 35, 241–249 (1944)] of Milne's kinematic relativity. They show that Wilson's assumption that $\mathbf{V}t$ is the space component of a four-vector is false; this was also pointed out in the review of Wilson's paper [these Rev. 6, 72]. *A. Schild* (Toronto, Ont.).

Whitrow, G. J. On the vectors and invariants of kinematic relativity. Philos. Mag. (7) 36, 170–178 (1945). [MF 13339]

As in the papers by Newing and Milne reviewed above, Wilson's proof that Milne's equations of motion of a free particle are not Lorentz invariant is shown to be erroneous. Invariants of kinematic relativity, that is, invariants under the group of Lorentz rotations of space-time, are constructed. *A. Schild* (Toronto, Ont.).

Camm, G. L. The two-body gravitational problem in kinematical relativity. Nature 155, 754–755 (1945). [MF 12779]

Schild [Phys. Rev. (2) 66, 340–342 (1944); these Rev. 6, 241] has criticized Milne's gravitational equations on the ground that the "simultaneity" relation used to integrate them is not invariant under Lorentz transformations. It is now shown that there is one and only one invariant relation of this kind, namely,

$$t_1^2-\mathbf{P}_1^2/c^2=t_2^2-\mathbf{P}_2^2/c^2,$$

where $(t_1, \mathbf{P}_1)$, $(t_2, \mathbf{P}_2)$ are the epochs and position vectors of the two gravitating particles. However, whatever relation is adopted, the equations of motion possess an integral of energy. *G. C. McVittie* (London).

Whitrow, G. J. The two-body problem in Milne's theory of gravitation. Nature 156, 365–366 (1945). [MF 13530]

Camm has given the Lorentz-invariant relation $t_1^2-\mathbf{P}_1^2/c^2=t_2^2-\mathbf{P}_2^2/c^2$ linking the event $(t_1, \mathbf{P}_1)$ at a gravitating particle m_1 with the event $(t_2, \mathbf{P}_2)$ at another gravitating particle m_2 [see the preceding review]. It is now proved that the relation is equivalent to $\tau_1=\tau_2$ if the dynamical time $\tau=t_0\log(t/t_0)+t_0$ is used instead of the cosmical time t. This is said to furnish a conclusive reason for adopting Camm's relation. *G. C. McVittie* (London).

Patwardhan, G. K., and Vaidya, P. C. Relativistic distributions of matter of radial symmetry. J. Univ. Bombay (N.S.) 12, part 3, 23–36 (1943). [MF 12941]

Following Tolman [Phys. Rev. (2) 55, 364–373 (1939)], the authors consider the general static isotropic line-element

$$ds^2=-e^\lambda dr^2-r^2(d\theta^2+\sin^2\theta d\phi^2)+e^\nu dt^2, \tag{1}$$

where λ and ν are functions of r. The nonzero components of the energy-momentum tensor are T_1^1, $T_2^2=T_3^3$ and T_4^4. The condition $T_1^1=T_2^2$ for the stresses to reduce to a pure pressure p $(=-T_1^1)$ yields the "isotropy equation"

$$e^{-\lambda}(\tfrac{1}{2}\nu''+\tfrac{1}{4}\nu'^2+\tfrac{1}{2}(\nu'-\lambda')r^{-1}-\lambda'\nu'r^{-1})=e^{-\lambda}(\nu'r^{-1}+r^{-2})-r^{-2}.$$

A number of particular solutions are examined and in each case expressions are obtained for the pressure p and the density ρ $(=T_4^4)$. Next, the problem of finding the line-element (1) is investigated, given the density distribution

(that is, the function $\rho(r)$) and assuming as boundary conditions the continuity of the $g_{\mu\nu}$ and the vanishing of p at the surface of a sphere $r=a$. This is achieved by a method of successive approximations, which is carried out in detail to the third order in ρ. Finally, the energy of an isotropic distribution of matter within a sphere $r=a$ is discussed. It is shown that the energy is determined by the surface value of the $g_{\mu\nu}$ only in the case where the pressure at the surface is zero. *A. Schild* (Toronto, Ont.).

Mariani, Jean. Une interprétation théorique du magnétisme terrestre et solaire. C. R. Acad. Sci. Paris **218**, 585–586 (1944). [MF 13469]

From the generalized gravitational field equations proposed by the author [same C. R. **218**, 447–449 (1944); these Rev. **6**, 241], a relation between the magnetic moment and the angular momentum of a rotating homogeneous sphere is deduced. The result is applied to the earth and the sun; the calculated magnetic fields are found to be in good agreement with observation. *A. Schild* (Toronto, Ont.).

Eddington, Arthur S. The combination of relativity theory and quantum theory. Communications Dublin Inst. Advanced Studies. Ser. A. [Sgríbh. Inst. Árd-Léigh. Bhaile Átha Cliath] no. **2**, 69 pp. (1943). [MF 12886]

The author points out the need for a relativistic theory of probability distribution. Such a theory is developed and applied to the theoretical calculation of all the fundamental physical constants except the cosmical number. The article is divided into four parts: (I) The uncertainty of the reference frame, (II) Multiplicity factors, (III) Electrical theory, (IV) Gravitation, exclusion and interchange.

In part I, the author maintains that only the relative coordinate of two physical entities is an observable. When more than two particles are considered it may not be convenient to identify the origin with one of the particles. A sharply defined geometrical origin would be contrary to the uncertainty principle. Hence a "physical origin" is introduced, which possesses a Gaussian probability distribution with respect to the geometrical frame. With the aid of Bernoulli's theorem the standard deviation σ of the distribution is found to be $\sigma=R_0/2N$, where R_0 is the radius of spherical 3-space, and N is the number of particles in the universe.

A quantity whose value is given as free information (for example, charge and mass) is called a stabilised characteristic. From this concept, a general theory is developed according to which stabilisation corresponds to the introduction of constraints in the phase space of the particle. In part II, this idea is developed further in connection with the "self-consistent field." The theory is applied to the calculation of the mass ratio of the proton and electron.

Part III introduces a correction for electrical energy in the mass ratio calculation. It is pointed out that mass, momentum and charge are defined in quantum theory to apply to quantities which are the analogues of the quantities in macroscopic physics. The quantum units are fixed so that the masses of neutral atoms take the same value in both theories; but then it is found that the quantum value of the charge e is not equal to the analogous macroscopic charge. This difficulty accounts for the discrepancy between the spectroscopic and deflection methods of measuring the ratio e/m. Part IV contains the deduction of the gravitational constant as well as a calculation of the non-Coulombian energy of two protons. *T. F. Morris and A. Schild.*

Biben, Georges. Le dualisme "ondes-corpuscules" et la démonstration de l'identité entre le principe de Fermat et le principe de Maupertuis. J. Math. Pures Appl. (9) **22**, 55–69 (1943). [MF 12250]

This paper concerns the unification of general relativity and wave mechanics. Postulating a second order partial differential equation of hyperbolic type (the wave equation), the author seeks to interpret the bi-characteristic curves associated with the equation as the trajectories of particles given by relativity theory. Basing his considerations upon the forms of the differential equations defining the bi-characteristic curves, he obtains various results which are formally similar to Lagrange's equations, Hamilton's canonical equations, the principle of least action, and other features of dynamical theory. An overly elaborate symbolism and many typographical errors make it difficult to follow the details of the argument. Little is given in the way of an explicit discussion of the physical meaning of the results. It appears from the author's own remarks that he has made much use of the works of other writers, and the reviewer remains uncertain as to just what parts of the contents of the paper are really new. *L. A. MacColl.*

MECHANICS

Dubin, Charles. La représentation des rayons de giration par un cercle d'inertie. Génie Civil **119**, 233–234 (1942). [MF 13799]

Mouette. Étude de l'ellipse centrale d'inertie d'un triangle. Ann. Ponts Chaussées **1941** I (111ᵉ année), 361–372 (1941). [MF 12884]

Hulubei, Dan I. Sur un problème de statique. Bull. Math. Soc. Roumaine Sci. **42**, no. 1, 3–13 (1940). [MF 12733]

Dans ce travail nous traitons le problème des réactions d'un plan horizontal sur lequel un corps pesant s'appuie par n points. Cela nous amène, tout naturellement, à faire quelques remarques sur les domaines convexes. *Extract from the paper.*

Hulubei, Dan I. Sur l'annulation de la réaction dans le cas du pendule sphérique. Bull. Math. Soc. Roumaine Sci. **42**, no. 1, 15–18 (1940). [MF 12734]

A treatment of the elementary problem of determining the conditions under which the tension in the supporting string in a simple spherical pendulum may vanish. *D. C. Lewis* (Durham, N. H.).

Stoenescu, A. Sur les systèmes de forces sous l'action desquelles un mobile parcourt une trajectoire donnée. Bull. École Polytech. Bucarest [Bul. Politehn. București] **13**, 309–315 (1942). [MF 13568]

The author finds a differential equation to be satisfied by the components of a force (assumed to be functions of position not depending explicitly on the time), if a particle, to which the force is applied, is to move along a prescribed

trajectory. For the complete determination of the two components of force, another equation, involving the coordinates of the particle and the components of the force, must be given, as well as the velocity of the particle at a given point on its prescribed trajectory. Several examples are considered. *D. C. Lewis* (Durham, N. H.).

Stoenescu, A. Sur les systèmes de forces sous l'action desquelles un mobile parcourt une trajectoire donnée. Bull. École Polytech. Bucarest [Bul. Politehn. Bucureşti] 14, 157–162 (1943). [MF 13577]

This paper generalizes to three dimensions the results reviewed above. *D. C. Lewis* (Durham, N. H.).

Ţiţeica, Gabriela. Détermination par une méthode graphique des rayons de courbure des trajectoires initiales d'un certain système de points en mouvement. Bull. École Polytech. Bucarest [Bul. Politehn. Bucureşti] 14, 163–169 (1943). [MF 13578]

Soient n points matériels, de masses égales m, liés entre eux par des fils parfaitement flexibles et inextensibles, de poids négligeable et ayant tous la même longueur l. Supposons que ces points forment un polygone régulier et que leur mouvement est une rotation uniforme de vitesse angulaire ω autour du centre de gravité de ce polygone. Dans ce qui suit, nous voulons déterminer par une méthode graphique les rayons de courbure des trajectoires initiales de ces points, quand l'un des fils casse brusquement. *Extract from the paper.*

Sikorski, G. Sur la forme, que prend un fil d'égale résistance animé d'un mouvement de rotation autour d'un axe fixe. Bull. École Polytech. Bucarest [Bul. Politehn. Bucureşti] 14, 19–26 (1943). [MF 13573]

Karas, Karl. Allgemeine Zerlegung der Beschleunigung des komplan bewegten starren ebenen Systems. Z. Angew. Math. Mech. 24, 83–86 (1944). [MF 13194]

In plane motion with angular velocity ω and angular acceleration ϵ, for M an arbitrary and N a related fixed point in the moving plane, the acceleration of any point A may be decomposed into $AM\omega^2$ along AM, and $NA\epsilon$ perpendicular to NA. A graphical method of locating N is given. *P. Franklin* (Cambridge, Mass.).

Bateman, H. The derivation of Euler's equations from a variational principle. Revista Ci., Lima 47, 111–117 (1945). [MF 12710]

The author derives Euler's equations without using Euler's angles, by a variational method which is also applicable to some related problems on the free motion of a rigid body. *P. Franklin* (Cambridge, Mass.).

Benedikt, E. T. Erratum: On the representation of rigid rotations. J. Appl. Phys. 16, 551 (1945). [MF 13380]

The paper appeared in the same J. 15, 613–615 (1944); these Rev. 6, 23.

Dugas, René. Le point de vue de Jacobi en mécanique analytique classique et ses prolongements modernes. Rev. Sci. (Rev. Rose Illus.) 78, 345–347 (1940). [MF 13316]

Giarratana, Joseph. Equations of motion for classical dynamical systems of variable mass. Phys. Rev. (2) 68, 130–141 (1945). [MF 13526]

The systems discussed consist of masses inside arbitrarily defined containers. Part of the masses may move in or out. Except for the explicit assumption that certain quantities have continuous first and second derivatives, the treatment contains few ideas or results not found in the usual discussion of rocket problems in mechanics or the conventional derivation of the equations of hydrodynamics. *P. Franklin* (Cambridge, Mass.).

Lévy, Paul. Explication élémentaire de l'effet gyroscopique. Bull. Sci. Math. (2) 65, 9–20 (1941). [MF 13267]

This attempt to give an intuitive explanation of gyroscopic phenomena, without use of the usual differential equations, is confined to the case where the inertia of the rotating body has spherical symmetry. *D. C. Lewis.*

Jefferson, H. Gyroscopic coupling terms. Philos. Mag. (7) 36, 223–224 (1945). [MF 13348]

Some remarks concerning electrical analogues of mechanical systems with gyroscopic couplings, suggested by a paper by A. Bloch [Philos. Mag. (7) 35, 315–334 (1944); these Rev. 6, 23]. *L. A. MacColl* (New York, N. Y.).

Wells, D. A. A "power function" for the determination of Lagrangian generalized forces. J. Appl. Phys. 16, 535–538 (1945). [MF 13378]

In many nonconservative dynamical systems, the Lagrangian forces can be expressed in the form $Q_i=\partial P/\partial\dot{q}_i$, where P is a function of the q's, $\dot{q}$'s, and t. A well-known classical example is the case in which P is a quadratic form in the $\dot{q}$'s, when P is known as the Rayleigh dissipation function. In the present paper more general forms of P are shown to be of practical importance. *D. C. Lewis* (Durham, N. H.).

Dobronravov, V. V. Integral invariants of the equations of analytical dynamics in nonholonomic coordinates. C. R. (Doklady) Acad. Sci. URSS (N.S.) 46, 179–181 (1945). [MF 12698]

The author obtains the various integral invariants for a dynamical system expressed in nonholonomic coordinates. He is correct insofar as he is concerned with the use of nonholonomic coordinates for holonomic systems. He is wrong, however, in his statement that these integral invariants are available also for systems with nonholonomic constraints. In fact, there is no reason to believe that a curve for which $\delta\pi_i=0$ $(i=1,\cdots,m)$ should not, in general, flow into a curve for which $\delta\pi_i\neq0$. Moreover, C. J. Blackall has supplied a counterexample [Amer. J. Math. 63, 155–168 (1941); these Rev. 2, 206]. *D. C. Lewis* (Durham, N. H.).

Dramba, Constantin. Le problème plan de Lagrange et les chocs triples imaginaires. Bull. Math. Soc. Roumaine Sci. 43, 3–6 (1941). [MF 12724]

In the three body problem, there is a triple imaginary collision if the three mutual distances vanish at the real or imaginary instant t_0 but not all the corresponding coordinate differences vanish. The present note deals with triple imaginary collisions in the case in which the bodies move in a fixed plane so that they form, at each instant, the vertices of an equilateral triangle. If x and y are the rectangular coordinates of one of the bodies with respect to

the center of mass of the system, these coordinates are expressible as series proceeding in nonnegative integral powers of $(t-t_0)^{\frac{1}{3}}$. The solution in the neighborhood of a triple imaginary collision depends upon three arbitrary constants, namely t_0, the value of x at the collision and the energy constant. *L. A. MacColl* (New York, N. Y.).

Astronomy

Samoilova-Yachontova, N. The differential correction of elliptic orbits. Bull. Inst. Astr. Acad. Sci. URSS no. 53, 447–455 (1944). (Russian. English summary) [MF 12899]

This paper deals with the problem of obtaining corrections to elliptical orbits of planets, asteroids or comets when several pairs of corrections to the spherical coordinates of the object are known. Eckert and Brouwer [Astr. J. 46, 125–132 (1937)] published a new method adapted to machine computation, in which the rectangular corrections δx, δy, δz are expressed as functions of six quantities which involve the corrections to the orbital elements but are immediately available if the rectangular coordinates and velocities are known, as they often are. The author has derived a new set of equations of condition relating the values of δx, δy, δz directly to the corrections of the elements δM_0, δi, $\delta\Omega$, $\delta\omega$, $\delta a/a$ and δe. She claims that these formulae, also adapted to machine computation, are somewhat more convenient than those given by Eckert and Brouwer for the case of small eccentricities. The latter are preferable when the eccentricities are not too small and when it is not necessary to obtain the elements themselves but only the quantities defined by Eckert and Brouwer. She has given a numerical example for the asteroid (287) Nephtys. *O. Struve* (Williams Bay, Wis.).

Bagenov, G. The law of areas in the disturbed motion of an asteroid. Astr. J. Soviet Union [Astr. Zhurnal] 21, 170–175 (1944). (Russian. English summary) [MF 12332]

v. Weizsäcker, C. F. Über die Entstehung des Planetensystems. Z. Astrophys. 22, 319–355 (1943). [MF 11858]

This paper presents a theory of the formation of the solar system from a gaseous envelope rotating with the sun. The implications of the theory are analyzed in considerable detail and physically plausible arguments are given to substantiate the conclusions. It is shown that, for a stable motion of the molecules of such a gas under the gravitational attraction of the sun, the density would decrease exponentially from the plane of solar rotation. A description is given of a process by which viscosity effects could lead to formation of concentric bands of "vortices"; each band contains several symmetrically placed vortices; each vortex consists of a family of molecules moving on Kepler ellipses with common major axis and period. It is then proposed that the planets were formed as the result of condensation effects along the surfaces of contact of adjacent bands. In particular, this leads to a theoretical explanation of the Titius-Bode rule for mean planetary distances from the sun. In the condensation process the heavier elements would appear first, while the lighter elements would eventually fly off to interstellar space; this is consistent both with the conservation of angular momentum of the system and with the contrasting chemical compositions of the planets, on the one hand, and the sun, on the other. A critical review by G. Gamow and J. A. Hynek has appeared in Astrophys. J. 101, 249–254 (1945). *W. Kaplan.*

Parijsky, N. On the origin of the solar system. The solution of Russell's problem. Astr. J. Soviet Union [Astr. Zhurnal] 20, no. 2, 9–29 (1 plate) (1943). (Russian. English summary) [MF 12888]

Parijsky, N. N. On the origin of the solar system. II. Supplements to the classification of the orbits in the restricted hyperbolic problem of three bodies (Russell's problem). Astr. J. Soviet Union [Astr. Zhurnal] 21, 69–79 (1944). (Russian. English summary) [MF 12331]

These papers were inspired by a remark of H. N. Russell that the observed angular momenta of the planets per unit mass are too large to be explained as the result of the attraction of another star whose close approach to the sun was thought by Jeans and by Jeffreys to have produced the planetary system. The author has set himself the task of computing, by numerical integration, the equations of motion of a planet of infinitesimal mass which is assumed to have left the surface of the sun with a specified initial velocity, in a radial direction toward the perihelion point of the orbit of the passing star. The mass of this passing star is assumed to be equal to that of the sun, its perihelion distance is taken to be three times the radius of the sun, and its velocity at infinity with respect to the sun is assumed to be 30 km./sec. (this would make the eccentricity of the hyperbolic orbit equal to 1.007028). The physical mechanism required to project the planet with the specified velocity is not defined.

In the first paper the author integrates several cases with initial velocities of the planet ranging from 0.41 to 1.25 times the parabolic velocity with respect to the sun. He finds that for $v_{p1}<0.60\,v_{par}$, the planet falls back into the sun. For $v_{p1}=0.75\,v_{par}$, the planet is captured by the passing star. For $v_{p1}>0.75\,v_{par}$, the planet at first revolves around the sun in orbits which are of the order of 10 times the radius of the sun. As v_{p1} is increased the orbits become more elongated and the planet again falls into the sun. Finally, for very large values of v_{p1} the planet recedes to infinity.

In the second paper the author integrates the case $v_{p1}=0.6543\,v_{par}$. The computations were carried out for values of the radius vector of the passing star from 1.00 to 4.47. The result is a new class of planetary orbits around the sun, of very small dimensions and of small angular momentum per unit mass. The planet moves in a complicated orbit, but the author computes several sets of osculating elliptical elements and shows that the final orbit is almost circular with a semidiameter of about 2 or 3 times the radius of the sun.

In all cases thus far investigated by the author, the ratio of the angular momentum per unit mass of the resulting planet to that of the passing star is less than one. He concludes that the computations support Russell's contention, since the angular momentum of the passing star per unit mass must have been much less than that of even the planet Mercury, for any value of the perihelion distance which is sufficiently small to produce violent disturbances in the surface layers of the sun. *O. Struve.*

García, Godofredo. On the occurrence in the solar system of dissipative and gyroscopic forces in addition to universal gravitation. Revista Ci., Lima 47, 173–273 (4 plates) (1945). (Spanish) [MF 13370]

The program of several previous papers of the author [same Revista 46, 507–584 (1944); Actas Acad. Ci. Lima 8, 3–6 (1945); these Rev. 6, 190] is continued. The prob-

lem of three bodies subject to gravitational, dissipative and gyroscopic forces is considered in detail. The homothetic solutions are studied, libration points are determined and their stability analyzed. *W. Kaplan.*

García, Godofredo. Generalization of the Lagrange-Birkhoff inequality for a dissipative system in an approximately Newtonian field. Actas Acad. Ci. Lima 7, 431–434 (1944). (Spanish) [MF 12091]

This is essentially the same as a previous paper by the author [Revista Ci., Lima 45, 281–292 (1944); these Rev. 6, 190]. *W. Kaplan* (Ann Arbor, Mich.).

Chandrasekhar, S. Stellar dynamics. Monthly Not. Roy. Astr. Soc. 105, 124–134 (1945). [MF 13283]

The first problem discussed is that of stellar encounters pictured in terms of the two-body problem. The change of velocity produced by a single collision is relatively small, hence encounters produce an effect only as they become numerous. The analogy is therefore with Brownian motion and not with the kinetic theory of gases. As in the case of Brownian motion, the effect of stellar encounters may be represented as a process of diffusion in the velocity-space. The theory is applied to the determination of the velocity of escape of a star from the Pleiades cluster and yields a mean life for the cluster of 3×10^9 years.

The second problem is the dynamical representation of the observed stellar kinematics of the galaxy. It can be proved that, in this connection, stellar encounters are unimportant. The observed motions can be analyzed into a field of differential motions together with the phenomenon of star streaming, which lead to a general form of velocity distribution function. The dynamical problem is then to solve the equation of continuity for this function and simultaneously to deduce the gravitational potential function. The following special problems have been solved: stellar systems (1) in a steady state and of finite extent, (2) in a nonsteady state and with circular symmetry, (3) in a nonsteady state and with a spherical distribution of residual velocities.

A third question relates to the theory of multiple collisions, which is reducible to the analysis of the fluctuating gravitational field to which a star is subjected owing to the changing complexion of the local stellar distribution. The theory is applied to discover the time of dissolution of binary systems in the galaxy and again leads to a time-scale which cannot be any large multiple of 3×10^9 years.

Lastly a brief reference is made to Lindblad's theory of the spiral structure of nebulae. *G. C. McVittie.*

Fricke, W. Über die Relaxationszeit in Sternsystemen. Z. Astrophys. 20, 268–277 (1941). [MF 12927]

This paper criticizes an evaluation of the time of relaxation of stellar systems by Chandrasekhar [Astronomical Papers Dedicated to Elis Stromgren, pp. 1–24, Einar Munksgaard, Copenhagen, 1940; these Rev. 3, 216]. [The paper was written before the details of the calculations were published by Chandrasekhar in Astrophys. J. 93, 285–304 (1941).] *S. Chandrasekhar* (Williams Bay, Wis.).

Milne, E. A. The natural philosophy of stellar structure. Monthly Not. Roy. Astr. Soc. 105, 146–162 (1945). [MF 13284]

In this address to the Royal Astronomical Society the author raises certain objections against the current (and more specifically Eddington's) treatment of the problem of stellar equilibrium. His objections are in the main derived from his critique of the "standard model." More particularly, if the coefficient of opacity κ and the rate of generation of energy ϵ per gram of the stellar material are assumed constant through a gaseous star, then (in a standard notation)

$$L=4\pi CGM(1-\beta)/\kappa$$

and

$$1-\beta=\text{constant}\times M^2\mu^4\beta^4.$$

Milne believes that, since "the atomic properties which control ϵ $(=L/M)$ and those which control κ are so essentially different," "if the original standard model tells us anything at all, it tells us that in general wholly gaseous configurations will not be found in nature."

Milne's second objection is derived from Eddington's treatment of the model $\kappa\eta=$constant. Eddington writes

$$L=4\pi cGM(1-\beta)/\kappa_c\bar{\eta}, \tag{1}$$

where κ_c is the opacity at the center of the star and $\bar{\eta}$ is a constant to which a value in the neighborhood of 2.5 is generally assigned. This, according to Milne, is an error and it is not possible to infer the interior opacity in this or any other manner. Instead, Milne replaces equation (1) by

$$L=4\pi cGM(1-\beta_0)/\kappa_0,$$

where β_0 and κ_0 refer to the "boundary values."
S. Chandrasekhar (Williams Bay, Wis.).

Hoyle, F. On the integration of the equations determining the structure of a star. Monthly Not. Roy. Astr. Soc. 105, 23–29 (1945). [MF 12805]

The paper discusses the most advantageous way of integrating the equations of stellar equilibrium. [The author states that only very few "inward integrations" of the stellar equations exist. Actually, the reviewer and his students have at their disposal over thirty integrations suitable for use under a variety of conditions.]
S. Chandrasekhar (Williams Bay, Wis.).

Hydrodynamics, Aerodynamics, Acoustics

San Juan Llosa, Ricardo. Some questions in fluid mechanics. Revista de Aeronáutica. Madrid, no. 42, 16 pp. (1944). (Spanish) [MF 12781]

The paper contains general considerations concerning the interdependence of a stationary character of the velocity, density and pressure of a fluid. It discusses the conditions for the existence of an acceleration potential and gives a definition of the density on the basis of the concept of generalized convergence of G. Birkhoff. *I. Opatowski.*

Cârstoiu, I. Nouveaux points de vue sur quelques théorèmes fondamentaux de la mécanique des fluides. Bull. École Polytech. Bucarest [Bul. Politehn. Bucureşti] 13, 42–45 (1942). [MF 13565]

Cârstoiu, I. Sur le mouvement général d'un fluide parfait. Bull. École Polytech. Bucarest [Bul. Politehn. Bucureşti] 13, 316–323 (1942). [MF 13569]

★Sauer, Robert. Theoretische Einfuhrung in die Gasdynamik. J. W. Edwards, Ann Arbor, Mich., ~~1943~~. 1945 vii+146 pp. $4.50.

[Photographic reprint of the original edition published by Springer, Berlin, 1943.] This short monograph deals only

with the mathematical aspects of the theory of compressible fluids but is intended primarily for engineers and physicists. Accordingly, several subtler mathematical questions arising in gas dynamics are omitted. Nevertheless, the reader will find a clear presentation of the mathematical methods used up to about 1940 for the treatment of steady adiabatic flows of a perfect compressible fluid (unsteady motions are not discussed).

The first chapter contains the derivation of the equations of motion and the discussion of the fundamental concepts of gas dynamics. Simple typical flows are described. The second chapter is devoted to the Prandtl-Glauert linearization, that is, to the theory of almost uniform flows. Both subsonic and supersonic flows are treated, including flows around obstacles, especially around airfoils and bodies of revolution of arbitrary shapes. Two special but very typical flows are the subject of the next chapter: the two-dimensional flow past a corner and the rotationally symmetrical flow around a circular cone, the discussion being centered around the concept of shocks. Chapter 4 treats the difficult problem of integrating the actual nonlinear equations of motion. The first section describes the various schemes of successive approximations: the Rayleigh-Janzen expansion of the potential function in a power series in the stream Mach number, a modification of this method due to Prandtl and the solution of mixed (transonic) flows by means of power series in the space variables. A rather sketchy account of the hodograph method contains the linearization of the equations of motion by means of the Legendre transformation and by means of the Molenbroek-Chaplygin transformation. The Chaplygin-Kármán-Tsien linearization of the pressure-volume relation is presented, as well as Ringleb's method of obtaining rigorous solutions of the compressibility equations. [Ringleb's method is, of course, essentially a restatement of part of Chaplygin's work. Chaplygin's solution of the jet problem is not even mentioned.] A complete account is given of the Prandtl-Busemann method for constructing two-dimensional supersonic flows with and without shock lines, and the method is extended to rotationally symmetrical flows. Finally, the equations of motion for the case of nonvanishing vorticity are derived.

The book contains many graphs and several numerical tables. Graphical methods are consistently stressed. References to literature in German and English are rather complete; almost no account is given, however, of the important work done in Russia. *L. Bers* (Syracuse, N. Y.).

Pavel, D. Strömungsbilder für einzelne oder mehrere Brunnen. Bull. École Polytech. Bucarest [Bul. Politehn. Bucureşti] 12, 242–248 (1941). [MF 13558]

Silber, Robert. Sur une mécanique des fluides compressibles basée sur le remplacement du champ de vitesse par le champ de quantité de mouvement. C. R. Acad. Sci. Paris 220, 162–164 (1945). [MF 13492]

The author tries to simplify the problem of flow of a nonviscous compressible fluid by substituting rot $\rho\mathbf{V}=0$ for rot $\mathbf{V}=0$; ρ is the density and $\mathbf{V}$ the velocity vector. The author's "justifications" for this innovation are irrelevant, since he has not disproved the validity of Kelvin's well-known theorem which requires rot $\mathbf{V}=0$ for flow of a compressible fluid if the flow is irrotational at any one instant and if the pressure is a function only of density (isentropic flow). *H. S. Tsien* (Pasadena, Calif.).

Carafoli, E. Sur la détermination des caractéristiques aérodynamiques des profils déformés. Bull. Math. Phys. Éc. Polytech. Bucarest 11, 155–161 (1940). [MF 13549]

If the equation transforming a circle in the ζ-plane into an airfoil in the z'-plane is $z'=z'(\zeta)$, the transformation of the same circle into a deformed profile may be written $z=z'+\sum q_n'/\zeta^n$. The coefficients q_n' are determined approximately by a Fourier analysis of the given deformation function, and the lift and moment of the deformed profile are then calculated. [This is similar to the method of von Kármán and Burgers in Aerodynamic Theory, vol. 2 (edited by W. F. Durand), Springer, Berlin (1935).] This method is applied to calculate the lift and moment of (1) a circular-arc profile of zero thickness, as a deformation of a flat plate, and (2) a Joukowski profile with deflected flap, as a deformation of the original profile. *W. R. Sears.*

Golubeva, O. V. Determination of lift on aerofoil upon separation of flow from its surface. C. R. (Doklady) Acad. Sci. URSS (N.S.) 46, 99–101 (1945). [MF 12705]

The case considered is that in which the alternate shedding of vortices from the trailing edge and from a point of separation on the upper surface produces a vortex street. An expression is derived for the circulation, assuming that the velocity at the trailing edge is always finite; this is simplified by an approximation in the summation over the wake vortices, said to be valid for practical angles of incidence and cambers. The mean value of the circulation is given by the expression

$$\Gamma_{\text{mean}}=\Gamma_0-f(\alpha)\Delta\Gamma,$$

where Γ_0 is the circulation if separation were not present, $\pm\Delta\Gamma$ is the strength of the wake vortices, and $f(\alpha)$ is a function only of the angle of attack. It is proposed to calculate the vortex-street breadth and the mean circulation by a process of successive approximations. First, the separation point is to be determined from boundary-layer theory using the pressure distribution for flow without separation. Assuming that the vortex-street parameters are known, one can then calculate first approximations to $\Delta\Gamma$ and Γ_{mean}. For an improved approximation to the pressure distribution, these values will be employed, with a crude representation of the vortex wake. *W. R. Sears.*

Szabó, I. Die Strömung um eine Fläche von elliptischem Umriss. Ing.-Arch. 14, 351–373 (1944). [MF 13375]

The wing of elliptical planform in a parallel stream is treated under the assumption that the deviation of the wing surface from a plane is small. The wing is approximated by a vortex sheet lying in the plane; the problem is to determine the distribution of vortex strength so as to make the induced velocity at every point compatible with the given geometry of the wing.

First, neglecting the existence of a vortex wake, the author assumes a special form for the vortex distribution, namely, a series of polynomials in two variables suggested by Koschmieder [Math. Ann. 91, 62–81 (1924)] and obtains a solution in the form of a doubly-infinite set of equations. The integrated polynomials are brought into forms involving tabulated elliptic integrals. Considering next the actual case involving a vortex wake, the author assumes the form of the vortex loading corresponding to circulation about the wing, by analogy with the two-dimensional theory. Moreover, he evaluates the average induced velocity due to the wake at any spanwise station and uses it for all points at that station, as in the lifting-line approximation. The result

is an additional series of terms in the set of equations mentioned above. Formulae are derived for the lift, induced drag, and pitching moment, in terms of the coefficients to be determined from these equations. This theory is applied especially to the cases of the flat elliptic plate, the antisymmetrically twisted plate, and a certain cambered surface. In these cases (especially the first) the formulae reduce to simple forms. The results for the flat plate are compared with those of Kreines [Z. Angew. Math. Mech. **20**, 65–88 (1940); these Rev. **2**, 27], König and Kinner, and good agreement is found.

Proceeding to the case of the yawed elliptical wing, the author introduces skew Cartesian coordinates and modifies his previous equations accordingly. As an approximation, however, he carries over the contribution due to vortex wake without correction. The subsequent analysis is analogous to that for symmetrical flow. The flat plate is treated in detail and it is stated that the results are in good agreement with those of Kreines and of Hörner.

W. R. Sears (Inglewood, Calif.).

Lighthill, M. J. Two-dimensional supersonic aerofoil theory. Ministry of Aircraft Production [London], Aeronaut. Res. Committee, Rep. and Memoranda no. **1929** (7384 and 7571), 19 pp. (1944). [MF 12932]

The first-order theory of Ackeret [Z. Flugtechnik Motorluftschiffahrt **16**, 72–74 (1925); Tech. Memos. Nat. Adv. Comm. Aeronaut. no. **317**] is reviewed. The conditions accompanying an inclined shock wave, or a series of such waves, are then determined and it is shown that conditions at a point cannot in general be determined solely by the angle of the boundary (airfoil surface); this would be true only to the second order. The expansive, isentropic flow about an exterior angle is then treated, and the relations for the velocity, pressure, etc., in terms of the local slope of a smooth surface are obtained by a rather complicated limiting process, starting from a sequence of corners. The maximum turning angle due to a shock and the turning angle to produce subsonic conditions are calculated and tabulated for various initial Mach numbers.

A procedure is set up for the calculation of the pressures on an airfoil using the exact results of the paper, and certain quantities are tabulated to facilitate the process. It is shown that Busemann's (second-order) approximation [Conv. Sci. Fis. Mat. Nat., Rome, 1935 (Volta Congress), pp. 328–360] is adequate for the condition behind the shock wave at reasonably large Mach numbers. In an appendix it is shown that the Ackeret (first-order) approximation deviates appreciably even for small angles.

Detailed calculations of lift, drag, and moment as functions of incidence are carried out to illustrate the application of the theory. The final results are shown in comparison with the corresponding ones of Ackeret's theory; the difference does not appear large except in the case of the moment coefficient. *W. R. Sears* (Inglewood, Calif.).

Lighthill, M. J. The conditions behind the trailing edge of the supersonic aerofoil. Ministry of Aircraft Production [London], Aeronaut. Res. Committee, Rep. and Memoranda no. **1930** (7412), 8 pp. (1944). [MF 12933]

This is a sequel to the paper reviewed above. It is pointed out that even in an inviscid fluid a discontinuity surface must exist behind the trailing edge, since the streams coming from above and below the airfoil have different velocities, densities, temperatures, and entropies. The angle of this surface to the main flow is calculated by summing the results of the various compressive shocks and expansions undergone by the fluid on top and bottom and introducing the condition of equal pressures at the rear. This angle is found to be very small in practical cases and may change sign. The absence of downwash behind the airfoil suggests an investigation of momentum conditions in the flow; it is shown that, to the first order, all of the downwash occurs between the Mach waves leaving the leading and trailing edges on top and bottom. [The existence of this downwash in the two-dimensional case is in contrast to the analogous subsonic case, where the downwash is zero and the lift appears in the pressure throughout the fluid. The author's remark on this subject does not appear to be correct.] Proceeding to the second-order theory, it is shown that the shock waves must be parabolic in shape far from the airfoil.

W. R. Sears (Inglewood, Calif.).

Oudart, Adalbert. Écoulement plan supersonique. Ondes de choc obliques et ondes de détente. C. R. Acad. Sci. Paris **220**, 274–276 (1945). [MF 13502]

Stewart, H. J. Hydrodynamic problems arising from the investigation of the transverse circulation in the atmosphere. Bull. Amer. Math. Soc. **51**, 781–799 (1945). [MF 14087]

The author studies the stability of various arrangements of vortices on a rotating disc. Friction and vertical velocities are neglected. It is further assumed that the horizontal momentum does not vary with height and that the density is constant. These simplifications presumably do not affect the problem substantially. The vortex models are steady and have circular symmetry. It is first shown that a single row of equally spaced equal vortices such as must form in a shearing zone on the boundary of the westerlies is unstable, while for double rows stable configurations exist. In the limiting case of small distances between the vortices of one row one is led to Kármán's vortex streets behind two-dimensional bluff bodies. Since the width of the vortex street is not small compared to the earth's dimensions these results cannot be applied directly to the atmosphere. As a closer approach to reality the stability of a ring system of vortices is investigated with the result that 6 or fewer equally spaced anticyclonic vortices of equal strength form a stable configuration. The existence of a single additional cyclonic vortex at the center (pole) reduces the stability so that the maximum number of vortices which form a stable configuration decreases. These anticyclones would have a small westward velocity but it can be expected that the cyclonic shear north of the westerlies would compensate for it so that the system would actually be stationary. Finally, the velocity distribution in a polar vortex on a rotating sphere rather than a rotating disc is investigated and shown to deviate considerably from that on a disc, especially at lower latitudes because of the effect of the Coriolis force.

B. Haurwitz (Río Piedras, P. R.).

Shvetz, M. On the boundary layer of the atmosphere. C. R. (Doklady) Acad. Sci. URSS (N.S.) **45**, 111–115 (1944). [MF 12577]

The theory of a planetary turbulent boundary layer is complicated as compared with the normal case in aerodynamics by the presence of large Coriolis forces. The fundamental difficulty of the treatment of turbulent motion, finding an expression for the exchange coefficient, is thus still more involved here. The author develops a method

for computing the velocity or shearing stress distribution in the atmosphere, similar to the well-known Pohlhausen procedure in aerodynamics. A velocity distribution is assumed which is made up of a number of terms depending on the number of boundary conditions with unknown coefficients. This profile has to satisfy the boundary conditions and, in the mean, the equations of motion. Since the influence of the Coriolis forces is small close to the earth's surface, the author adopts here a logarithmic velocity distribution and an exchange coefficient as given by Kármán's similarity law. The velocity distribution adopted thus consists of a logarithmic term plus a polynomial of the third degree. *H. W. Liepmann* (Pasadena, Calif.).

Batchelor, G. K. On the hydrodynamic resistance. Commonwealth of Australia. Council Sci. Ind. Res. Division Aeronaut. Rep. no. **955**, 30 pp. (1944). [MF 12669]

With a view to applications to experimental set-ups such as wire gauze in flow fields, the author studies an idealized hydrodynamic resistance defined as a surface of zero thickness, across which there is a pressure drop of magnitude $\frac{1}{2}k\rho q^2 \sec\theta$, where q and θ are, respectively, the magnitude and the inclination to the normal to the resistance of the local flow velocity, ρ is the density of the fluid, and k is a characteristic constant of the system. The resistance may be kept fixed or moving in a predetermined manner by the effect of external forces. It is stated that such an idealized system may be used for the description of the flow of real fluids by means of the perfect fluid theory, just as the usual notions of sources, vortices, etc., are used. The local conditions of discontinuity across the resistance are examined and the general field problem is set up for the case of steady two-dimensional flow. Explicit results are worked out for cases when the flow is a small perturbation of uniform motion. In the case of steady flow, the results agree with those of Collar [Air Ministry [London], Aeronaut. Res. Committee, Rep. and Memoranda no. **1867** (3900) (1939)]. In the case of turbulence, the changes in intensity and scale across the resistance are given explicitly in terms of the resistance coefficient k. Experiments are suggested to test these results. *C. C. Lin.*

Horton, C. W., and Rogers, F. T., Jr. Convection currents in a porous medium. J. Appl. Phys. **16**, 367–370 (1945). [MF 12542]

The stability of a fluid layer heated from below was first investigated by Rayleigh. The authors apply Rayleigh's method to the case where the fluid is contained in a porous medium. Thus a new physical parameter, the permeability k, enters and the dynamic equations of motion become

$$u=-(k/\mu)p_x,\quad v=-(k/\mu)p_y,\quad w=-(k/\mu)(p_z+g\rho).$$

These equations are to be solved together with the standard continuity and heat transfer equations. The temperature θ is taken as $\theta=\beta z+\vartheta$; that is, the temperature is made up of a mean part plus a small perturbation ϑ. It is assumed that u, v, w and ϑ are proportional to $e^{ilx}e^{imy}e^{nt}$ and the stability question is thus reduced to an eigenvalue problem. Instability and hence convection currents exist if $n>0$. This is found to occur for negative temperature gradients exceeding a certain critical value. The result is applied to a specific geophysical problem of diffusion through a sand layer. The numerical agreement is not too good.

H. W. Liepmann (Pasadena, Calif.).

Kalinin, N. K. On the solution of problems of ground-water motion by the method of P. J. Polubarinova-Kotschina. C. R. (Doklady) Acad. Sci. URSS (N.S.) **45**, 102–105 (1944). [MF 12575]

The paper concerns a steady plane flow of an incompressible fluid through an incompressible ground according to Darcy's law. The flow takes place in a region bounded by a prism of arbitrary polygonal cross section. The mathematical problem consists in finding a complex potential $f(z)$ satisfying certain boundary conditions. If the region of flow is mapped conformally into the upper half of a t-plane, the functions dz/dt and df/dt are solutions of a homogeneous linear differential equation of the second order whose coefficients are functions of t. The author investigates the singular points of this equation. *I. Opatowski.*

Poloubarinova-Kochina, P. J. Concerning unsteady motions in the theory of filtration. Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] **9**, 79–90 (1945). (Russian. English summary) [MF 13506]

The author considers a region of a porous medium occupied by two different liquids and containing sources of one of them. The boundary between the two liquids is a cylindrical surface S of infinite length. The change of the shape of S is investigated as the filtration goes on. Let C be a cross section of S and z the complex variable in the plane of C. It is assumed that the filtration flow admits a nonstationary logarithmic velocity potential F whose value on C is independent of the time t. This leads to a boundary condition $(\text{grad } F)^2=mdF/dt$, where m is a constant dependent on the porosity of the medium. The author considers first a single source inside an arbitrary C. Let $z=\sum A_iZ^i$ be a function mapping C conformally onto a circle. The problem consists of determining the functions $A_i(t)$ knowing their initial values. The complex potential is of the type $q\log f(Z)$, where q is a constant and $f(Z)$ a linear fractional function of Z. The author discusses in detail the case in which C is symmetric with respect to the real axis and the image of the source in the Z-plane lies on that axis. She expands $f(Z)$ in a Fourier series in $\arg Z$; this leads to a system of equations for the $A_i(t)$ of the type $\sum k_{ij}A_idA_j/dt=k$, where the k's are constants. She gives power series expansions for $A_i(t)$ in the case of an initially circular contour C and extends this result to the case when n sources are situated on a circle inside C and concentric with C.

I. Opatowski (Chicago, Ill.).

Leibenson, L. S. Turbulent movement of gas in a porous medium. Bull. Acad. Sci. URSS. Sér. Géograph. Géophys. [Izvestia Akad. Nauk SSSR] **9**, 3–6 (1945). (Russian. English summary) [MF 12581]

The gas flow in a porous medium is studied under the hypothesis of an isothermal regime, the gas being subjected to the law of Boyle-Mariotte. The differential equation of such a flow, with given initial and boundary conditions, can be integrated only by approximate methods.

E. Kogbetliantz (New York, N. Y.).

Leibenson, L. S. General problem of the movement of a compressible fluid in a porous medium. Bull. Acad. Sci. URSS. Sér. Géograph. Géophys. [Izvestia Akad. Nauk SSSR] **9**, 7–10 (1945). (Russian. English summary) [MF 12582]

The flow of a compressible fluid in a porous medium is studied in the most general case and its equation is formed. It is pointed out that the general solution with given initial and boundary conditions is very difficult. Some particular

integrals give the solution of the practically important problem of pressure fall at the opening of a porous layer.

E. Kogbetliantz (New York, N. Y.).

Koussakov, M. Theory of the method of measuring the viscosity of liquids by blowing off a film in a narrow wedge-shaped slit. Acta Physicochim. URSS **20**, 47–60 (1945). [MF 13314]

If a liquid possesses anomalous viscosity, that is, the value of viscosity depends on the velocity gradient, then the ordinary capillary viscometer is not very practical due to the large number of measurements necessary. Derjaguin, Koussakov and Krim [same Acta **20**, 35–46 (1945)] suggested a new method where air is blown through the wedge-shaped space formed by two plates wetted with the liquid to be investigated. The air velocity is parallel to the edge of the wedge. Due to the very thin liquid film, the shearing stress of the air on the liquid film can easily be calculated. It is shown that, if the motion is laminar, the shearing stress is very nearly proportional to the distance from the edge of the wedge. It is then deduced that the thickness of the liquid film is also proportional to the distance from the edge provided the viscosity of the liquid is independent of the velocity gradient. Then the interference fringes formed by the film will be straight lines meeting at one point on the edge of the wedge. If the liquid has anomalous viscosity, the fringes will be curved lines. By proper interpretation of the fringes, the viscosity of the liquid at different velocity gradients can be obtained at once.

H. S. Tsien.

Korn, Arthur. On vibrational vortices. Publ. Inst. Mat. Univ. Nac. Litoral **5**, 157–163 (1945). [MF 13726]

Jeffreys, Harold. Types of isostatic adjustment. Amer. J. Sci. **243-A** (Daly Volume), 352–359 (1945). [MF 12508]

Consider a semi-infinite medium and surface loading along a long strip. The loading is given by $H(x+l)-H(x-l)$, where $H(y)$ is the Heaviside unit function. If the medium is a viscous fluid, the discontinuities at $x=\pm l$ persist for all time and the disturbance remains essentially local. If a thin viscous fluid layer is sandwiched between strong upper and lower layers, the same type of loading smooths out according to erf $(x/t^{\frac{1}{2}})$, approximately. The mathematical analysis consists of substitution of the load function into known general solutions. The major part of the paper is devoted to discursions on general theories in geology and the implication of the results.

D. G. Bourgin.

Blokhintzev, D. Propagation of sound in a heterogeneous and moving medium. C. R. (Doklady) Acad. Sci. URSS (N.S.) **45**, 322–325 (1944). [MF 12662]

The author starts with the hydrodynamical equations of a compressible fluid, neglecting viscosity and heat conduction. The equations are linearized by assuming that (1) $\mathbf{v}$, p, ρ, S (S is the entropy) undergo small changes and higher powers of the increments are neglected. The contribution of the paper lies in the explicit inclusion of an entropy increment. For geometrical acoustics, as usual, the increments of the quantities in (1) are assumed to be harmonic functions, with amplitudes expansible in powers of the frequency. The author gives the equation, keeping linear terms in the frequency. For absorption, second degree terms in the frequency would be required also.

D. G. Bourgin.

Blokhintzev, D. I. Propagation of sound in turbulent flow. C. R. (Doklady) Acad. Sci. URSS (N.S.) **46**, 136–138 (1945). [MF 12701]

A turbulent flow is confined to a rectangular parallelepiped. The author resolves the velocity into micro and macro components in the sense of distance scale of variation. It is assumed that the macro component is constant and the micro component varies with the cube root of the period. The point of the author's contribution is the treatment of the micro aspect of the problem as a scattering problem. Thus one starts with a plane incident wave. An equation for the velocity potential for sound propagation in a moving medium is then used as the basis for a perturbation scheme. The solution is expressed as $\phi+\psi$, where ψ is the scattered component of the form $(B\exp i(\omega t-kR))/R$ for large R. The paper indicates the computation for the average of $|B|^2$, which is obviously connected with the absorption.

D. G. Bourgin (Urbana, Ill.).

Tunakan, Sadrettin. Eigenfrequenzen der Luftschwingungen in beiderseitig geschlossenen Röhren mit Blenden. Rev. Fac. Sci. Univ. Istanbul (A) **8**, 253–295 (1943). (German. Turkish summary) [MF 13600]

The author considers sound vibrations in a cylinder closed at both ends and containing diaphragms spaced equally along the length of the cylinder. The diameter of the cylinder is D; the diameter of the circular holes in the diaphragms is d; L is the distance between diaphragms. Using a theory due to Fouché [Rev. Fac. Sci. Univ. Istanbul, N.S. **3**, 285–310 (1938)], which is not described but is apparently of the transmission line variety, and estimating the drop in average pressure between the two sides of a diaphragm, he obtains the following set of equations when there are n diaphragms:

$$(k\pi D^2)^{-1}\tan k(L+\beta_i)$$
$$=d^{-1}-D^{-1}+(k\pi D^2)^{-1}\tan k(L+\beta_{i+1}),\quad i=0,1,\cdots,n,$$

where $k\beta_0=k\beta_n=\pi/2$. When β_i is eliminated from these equations, a set of equations results whose solution yields the allowed values of k. The author tabulates the solutions for $D/L=1$ for various values of d/D ranging from 0 to 1 for $n=1, 2, 3, 4$. The equations have the forms: $\cot\alpha=\frac{1}{2}K\alpha$, $\cot\alpha-\tan\alpha/2=K\alpha$, $\cot\alpha+\cot\alpha/2=K\alpha$, $2\cot\alpha\pm(2\cot^2\alpha+1)^{\frac{1}{2}}=K\alpha$, $3\cot\alpha+\cot\alpha/2\pm(5(\cot^2\alpha+1))^{\frac{1}{2}}=2K\alpha$, $3\cot\alpha-\tan\alpha/2\pm(5(\cot^2\alpha+1))^{\frac{1}{2}}=2K\alpha$.

H. Feshbach (Cambridge, Mass.).

MATHEMATICAL PHYSICS

Optics, Electromagnetic Theory

Bianu, B. Sur les miroirs paraboliques. Bull. École Polytech. Bucarest [Bul. Politehn. București] **13**, 46–51 (1942). [MF 13566]

Elegant derivation of the formula for longitudinal (spherical) aberrations for a parabolic mirror. The author studies the amount of retouching necessary to transform a spherical mirror into a useful parabolic mirror.

M. Herzberger (Rochester, N. Y.).

Bianu, B. Sur les miroirs paraboliques. II. Bull. École Polytech. Bucarest [Bul. Politehn. București] **14**, 51–58 (1943). [MF 13575]

[Cf. the preceding review.] The author calculates the transversal aberration of a point on and off the axis, comparing the ray through the center with a meridional ray of final aperture. He specializes his results for a telescope (small angular aperture) and investigates the image brightness.

M. Herzberger (Rochester, N. Y.).

Beutler, H. G. The theory of the concave grating. J. Opt. Soc. Amer. **35**, 311–350 (1945). [MF 12468]

A systematic theory of the image formation of a concave grating is developed. A great number of results of theoretical and practical interest are obtained by derivation from a basic principle of optics, Fermat's principle of least time. Let P be a point on the spherical mirror and w, l its x, y coordinates. If the separation of the grooves is denoted by d, light is essentially reflected only by the neighborhood of those points on the sphere for which w/d is an integer. A point B can be considered as a perfectly diffracted image of a point A if all the rays from A striking the surface at its reflecting parts reinforce each other at B. This is the case if the optical paths $AP+BP$ differ only by integral numbers of wave lengths λ when P wanders over the reflecting parts of the grating. On account of this the author characterizes the image formation of a grating by the function $F=AP+BP+(w/d)m\lambda$, the integer m being called the order of the image. The image is perfect if, for a certain arrangement of slit and plateholder, the characteristic function F remains constant. In practice this condition may be replaced by the Rayleigh condition that the variation of F shall not exceed the value $\frac{1}{4}\lambda$. The investigation is carried out by developing F into a power series with respect to w and l. With sufficient approximation this yields seven different contributions, representing different types of aberration from the ideal image which are related to the geometric aberrations of the mirror, spherical aberration, coma and astigmatism. The results of the general theory are used for a detailed discussion of different types of mountings (Rowland, Wadsworth, Eagle, etc.) and of their advantages or inherent defects. *R. K. Luneberg* (Buffalo, N. Y.).

Mooney, Robert L. An exact theoretical treatment of reflection-reducing optical coatings. J. Opt. Soc. Amer. **35**, 574–583 (1945). [MF 13414]

The problem of multiple reflection in certain types of low reflecting coatings is treated with the aid of Maxwell's electromagnetic equations. Instead of summing up the infinitely many single reflections on the boundaries of the layers, as is usually done in the literature of the subject, the author considers the situation as a boundary problem for plane waves. The boundary conditions follow from the fact that, on the layer surfaces, the normal components of the induction vectors and the tangential components of the field vectors must be continuous. The application of these conditions leads to a system of linear equations for the complex numbers which represent amplitude and phase of the different wave trains. The method is applied to the case of plane waves which are incident normally upon glass coated with one or two layers of nonconducting substances. Explicit formulas are derived for the reflectivity and transmissivity of such coatings and the results are discussed with regard to maximum and minimum reflectivity. In addition, formulas are given taking absorption into account.

R. K. Luneberg (Buffalo, N. Y.).

Ramachandran, G. N. On the radiation from the boundary of diffracting apertures and obstacles. Proc. Indian Acad. Sci. Sect. A. **21**, 165–176 (1945). [MF 12646]

When spherical waves, specified by a wave-function $u_0=r^{-1}e^{ik(r-ct)}$, are emitted from a source O and are incident on an opaque screen, the classical Kirchhoff-Helmholtz formula for the effect at a point P behind the screen is

$$(1)\qquad u=(4\pi)^{-1}\iint_\Sigma\left\{\frac{e^{ikR}}{R}\frac{\partial u_0}{\partial N}-u_0\frac{\partial}{\partial N}(e^{ikR}/R)\right\}dS,$$

where Σ is a surface bridging the aperture in the screen and r, R are the distances from O, P, respectively, to a typical point of Σ. If the distances from O and P to the screen are large compared with the wave length λ, (1) simplifies to the Fresnel formula

$$(2)\qquad u=-(\tfrac{1}{2}i/\lambda)\iint_\Sigma(rR)^{-1}e^{ik(r+R-ct)}(\cos\theta+\cos\theta')dS.$$

By a transformation due originally to Maggi (1888), but attributed here to Rubinowicz (1917), (1) can be written in the form

$$(3)\qquad u=\epsilon u_0-\int_\Gamma f ds,$$

where $\epsilon=0$ or 1 according as P is or is not in the geometrical shadow and Γ is the boundary of Σ, a formula which represents the diffracted light as due to sources on the boundary. In the present paper a formula of the same form as (3) is obtained under restrictive conditions.

The author assumes, although he does not say so clearly, that (i) the distances of O and P from the screen are large compared with λ, (ii) the plane of the aperture is effectively a large wave-front of radius D, (iii) θ and θ' are very small. Under these conditions he deduces from (2) the approximate formula

$$(4)\qquad u=-i\lambda^{-1}D^{-1}e^{ik(D-ct)}\iint_\Sigma R^{-1}e^{ikR}dS$$

(or rather its real part, since he does not use complex wave-functions). Following C. V. Raman (in his unpublished Sayaji Memorial Lectures), he proves that (4) can easily be turned into an expression of the form (3), a result which he could have got by approximating in Maggi's formula.

By the principle of stationary phase, he then shows that the dominant part of the line integral arises from a finite number of "poles" at which R is stationary, and deduces an approximation for the radii of the circular fringes formed inside the shadow of a circular disc. This idea has been applied in more detail by Y. V. Kathavate [same Proc. Sect. A. **21**, 177–210 (1945)], who gave geometrical constructions for the diffraction patterns of obstacles and apertures of various simple forms. *E. T. Copson* (Dundee).

Hutter, R. G. E. The electron optics of mass spectrographs and velocity focusing devices. Phys. Rev. (2) **67**, 248–253 (1945). [MF 12336]

The author derives the paths of electrons in an electromagnetic field of cylindrical symmetry as used in mass spectrometers and velocity focusing devices. The electrostatic potential φ is assumed to be a function only of r, the distance from the axis. The magnetic field is constant with the magnetic vectors parallel to the axis. The paths are derived with the aid of Fermat's principle and are given explicitly to the first and second orders. *R. K. Luneberg*.

Ambarzumian, B. On the problem of the diffuse reflection of light. Acad. Sci. URSS. J. Phys. **8**, 65–75 (1944). [MF 12337]

The author investigates the problem of diffuse reflection of light by a medium every volume element of which ab-

sorbs and reflects. The multiple reflections in the medium are described by a function $x(\theta, \varphi; \theta', \varphi')$, the "scattering indicatrix." This function determines the amount of light which is scattered by a volume element from the direction θ, φ into the direction θ', φ'. It is assumed that the indicatrix has the form $x=x(\cos\gamma)$, where $\cos\gamma=\cos\theta\cos\theta'+\sin\theta\sin\theta'\cos(\varphi-\varphi')$. The medium consists of plane parallel layers bounded on one side by a plane and extending to infinity on the other side. The problem is to find the diffuse reflection $r(\theta_0, \varphi_0; \theta, \varphi)$ of the total medium, that is, a function giving the angular distribution of the scattered light if a beam of parallel light (θ_0, φ_0) is incident on the boundary. With the aid of the fact that this function is unchanged if a finite layer is removed from the medium a quadratic integral equation is established for it. For the solution of this equation the author develops the indicatrix x and the unknown function r into series of Legendre polynomials. This allows him to replace the integral equation for r by a system of integral equations for certain universal functions $\varphi(\eta)$ of one variable. These equations are solved numerically for the case of a spherical indicatrix $x=1$ and for the case of an oblong indicatrix $x=1+x_1\cos\gamma$.

R. K. Luneberg (Buffalo, N. Y.).

Hurwitz, Henry, Jr. The statistical properties of unpolarized light. J. Opt. Soc. Amer. 35, 525–531 (1945). [MF 12874]

From a combination of physical and mathematical arguments the author concludes that the components of the field vector of a monochromatic unpolarized beam are of the form $E_x=X\cos\omega t+U\sin\omega t$, $E_y=Y\cos\omega t+V\sin\omega t$, where $X, \cdots, V$ are slowly varying functions of time which, at every instant, are mutually independent random variables with the same normal distribution. Hence the vector traces an ellipse with principal axes a and b so that $2ab\{a^2+b^2\}^{-1}$ is uniformly distributed between 0 and 1. *W. Feller.*

Doligez, M. Théorie corpusculaire de la lumière. Explication de l'entrainement des ondes par la matière en mouvement. J. Phys. Radium (8) 5, 136–141 (1944). [MF 12335]

From the result of the theory of relativity that light is deviated by a gravitational field the author concludes that one should expect a photon to travel through a transparent medium on a zigzag path. The deviations from a straight line are caused by passing through the strong gravitational field in the immediate neighborhood of atoms. The ratio of the length of such a path to its projection on its average direction determines the index of refraction of the medium. By an elementary and simple mathematical analysis of this hypothesis the author derives well-known formulae of Fresnel and Lorentz for the velocity of light in a moving medium. *R. K. Luneberg* (Buffalo, N. Y.).

Jones, R. Clark. A generalization of the dielectric ellipsoid problem. Phys. Rev. (2) 68, 93–96 (1945). [MF 12967]

Jones, R. Clark. Erratum: A generalization of the dielectric ellipsoid problem. Phys. Rev. (2) 68, 213 (1945). [MF 14112]

The author first solves the problem of determining the electric field inside an anistropic but homogeneous ellipsoid (with dielectric tensor ϵ and conductivity tensor σ) when it is embedded in a vacuum where there exists a uniform electric field $E_2e^{i\omega t}$. It is emphasized that such an external field will be possible only if the dimensions of the ellipsoid are small compared with the wave-length. The total polarization inside the ellipsoid is determined on the assumption of a uniform field $E_1e^{i\omega t}$ in the interior, and thence a tensor relation between E_1 and E_2 is obtained. A further generalization replaces the external vacuum by an anistropic medium with tensors ϵ_2 and σ_2. *M. C. Gray.*

Davy, N. The field between equal semi-infinite rectangular electrodes or magnetic pole-pieces. Philos. Mag. (7) 35, 819–840 (1944). [MF 12956]

Assuming the validity of the Laplacian differential equation, the field distribution is evaluated by conformal transformation. The transformation function, which involves elliptic integrals as well as elliptic functions, is evaluated in detail and certain equipotential surfaces and field lines are computed. Verification is obtained by exploring the field distribution in an experimental set-up. In addition, total charges, capacities and mechanical forces are evaluated.

E. Weber (Brooklyn, N. Y.).

Page, Leigh, and Adams, Norman I., Jr. Space charge between coaxial cylinders. Phys. Rev. (2) 68, 126–129 (1945). [MF 13525]

Craggs, J. W., and Tranter, C. J. The capacity of twin cable. Quart. Appl. Math. 3, 268–272 (1945). [MF 13538]

The problem of determining the capacity of two long parallel cylindrical conductors can be easily solved by the use of a conformal transformation. A simple extension of the method gives the result for the case in which each conductor is surrounded by a dielectric sheath whose boundary is a member of the coaxial system of circles defined by the boundaries of the conductors. The case in which the sheaths are concentric with the conductors is of much greater practical importance and in many types of cable the sheaths are actually touching. In this paper we give the derivation of the potential distribution for this latter case together with a practical method for the calculation of the capacity.

Extract from the paper.

Goddard, L. S. A method for computing the resonant wave-length of a type of cavity resonator. Proc. Cambridge Philos. Soc. 41, 160–175 (1945). [MF 12847]

In this paper a modification is developed of Hahn's method [J. Appl. Phys. 12, 62–68 (1941); these Rev. 2, 334] for calculating the electromagnetic field in the axially symmetric cavity resonator described in cylindrical coordinates (r, z, ϕ) by the inequalities $0\leqq r\leqq b$ if $-l<z<l$, $a\leqq r\leqq b$ if $-al\leqq z\leqq -l$ or $l\leqq z\leqq al$. The boundary conditions are the usual ones for a metallic boundary.

The cavity consists of a hollow cylinder A:

$$a\leqq r\leqq b, \quad -al\leqq z\leqq al$$

and a cylinder B: $0\leqq r\leqq a$, $-l\leqq z\leqq l$. The two parts have in common the cylindrical surface S: $r=a$, $-l\leqq z\leqq l$. In each of A and B the electromagnetic field components can be expanded in Fourier series with respect to z, the fundamental period being $2al$ for A and $2l$ for B. The condition of continuity of the field at S leads, after some manipulation, to an infinite system of linear equations in which the unknowns are the Fourier coefficients. In this

system expressions of the forms

$$T_n(\alpha)=\alpha^2 n^2 \sum_{m=1}^{\infty} R_m \frac{\sin^2(\pi m/\alpha)}{m^2-\alpha^2 n^2},$$

$$V_n(\alpha)=\alpha^2 n^2 \sum_{m=1}^{\infty} R_m \frac{m^2\sin^2(\pi m/\alpha)}{(m^2-\alpha^2 n^2)^2}$$

occur, in which R_m is an expression in Bessel functions of order 0 and 1. If $2al$ is smaller than the resonant wave-length, the variable of the Bessel functions is imaginary, the functions can be replaced by their asymptotic expansions and a series of descending powers of m is obtained for R_m. Thus T_n and V_n can be expressed in terms of the series $T_n{}^t$ and $S_n{}^t$ investigated in another paper [same Proc. 41, 145–160 (1945); these Rev. 7, 66] and the coefficients of the infinite system of linear equations can be computed from the tables given in that paper for $T_n{}^t$ and $S_n{}^t$. The diagonal coefficients are considerably larger than the non-diagonal ones and the equations may be rapidly solved by iteration. The resonant wave-length is tabulated for $a=1$, $b=2(2)8$, $\alpha=1(1)8$, 20, 50, 100, and $\alpha l=1, 2, 3$. An asymptotic formula for large α (narrow gap) is also obtained.

A. Erdélyi (Edinburgh).

MacLean, W. R. The reactance theorem for a resonator. Proc. I. R. E. **33**, 539–541 (1945).

The validity of Foster's reactance theorem stating the positivity of the slope of a driving point reactance function of a loss-free electrical network is proved for distributed electromagnetic systems such as cavity resonators, up to a frequency limit beyond which the concept of local driving point impedance becomes meaningless. The existence of such a limit region for any particular configuration precludes the usual analytical approach by means of meromorphic functions. The proof must rather be based on integral principles. The oddness of the reactance function follows from a well-known energy representation of the complex Poynting vector. The proof of the positivity of the slope is based on an extension of Helmholtz's theorem of adiabatic variations, the variation of frequency being attained by adiabatic motion of a shorting plug in a transmission line attached to the cavity while the system is oscillating.

H. G. Baerwald (Cleveland Heights, Ohio).

Adler, F. T. Three-dimensional Fourier transforms and their application to Maxwell's equations. J. Appl. Phys. **16**, 545–550 (1945). [MF 13379]

The author applies the finite Fourier transform to the solution of Maxwell's equations in rectangular coordinates. The boundary value problem of a rectangular cavity resonator with perfectly conducting walls is worked out in detail.

A. E. Heins (Cambridge, Mass.).

Preisman, Albert. Graphical analyses of nonlinear circuits. Quart. Appl. Math. **3**, 185–197 (1945). [MF 13531]

Gladwin, A. S. Energy distribution in the spectrum of a frequency modulated wave. I. Philos. Mag. (7) **35**, 787–802 (1944). [MF 12953]

Let

$$S(t)=\sin\left\{\Omega t+\sum_{m=1}^{q} b_m \sin(\omega_m t+\phi_m)\right\};$$

then it can easily be shown that

$$(1)\quad S(t)=\sum\left\{\prod_{m=1}^{q} J_{n_m}(b_m)\right\}\sin\left\{\left(\Omega+\sum_{m=1}^{q} n_m\omega_m\right)t+\sum_{m=1}^{q} n_m\phi_m\right\},$$

the summation being extended over all integral (positive, negative and zero) values of $n_1, \cdots, n_q$. [See Cherry and Rivlin, Philos. Mag. (7) 32, 265–281 (1941); these Rev. 3, 160.] If one assumes (as the author does) that the ω's are linearly independent ("unrelated," as the author calls them) then the right hand side of (1) is the Fourier series of $S(t)$ (in the sense which is used in the theory of almost periodic functions).

In applications of frequency modulation it is important to know how the energy of $S(t)$ is distributed over various portions of the spectrum. Exact formulas in terms of Bessel functions are much too cumbersome, but in certain limiting cases they can be greatly simplified. The limiting process assumed in this paper is the following. There exists a function $g(\omega)$ defined over an interval (α, β) $(\alpha>0)$, such that, as $q\rightarrow\infty$,

$$\sum_{m=1}^{q} b_m^2\psi(\omega_m)\rightarrow\int_\alpha^\beta g(\omega)\psi(\omega)d\omega$$

for every (Riemann integrable) function ψ. Moreover, as $q\rightarrow\infty$, the largest $|b_m|$ approaches 0. Under these assumptions the author shows how to calculate the energy, over a given frequency range, which is contributed by the frequencies of the form $\Omega+k\omega_i+l\omega_j$ $(i, j=1, \cdots, q;$ $k, l=0, \pm1, \pm2, \cdots)$. He claims that in various applications this contribution is a good approximation to the total energy in the range. An application to frequency modulation by telephone signals is discussed.

The reviewer would like to point out that it is easy to show that the total energy contained in the interval $(\Omega-\delta, \Omega+\delta)$ is proportional to

$$\pi^{-1}\int_{-\infty}^{\infty}\xi^{-1}\sin\delta\xi\prod_{m=1}^{q}J_0(2b_m\sin\tfrac{1}{2}\omega_m\xi)d\xi,$$

provided that neither $\Omega+\delta$ nor $\Omega-\delta$ belongs to the spectrum. In the limit considered by the author this expression becomes

$$\pi^{-1}\exp\left\{-\tfrac{1}{2}\int_\alpha^\beta g(\omega)d\omega\right\}\times\int_{-\infty}^{\infty}\xi^{-1}\sin\delta\xi\exp\left\{\tfrac{1}{2}\int_\alpha^\beta g(\omega)\cos\omega\xi d\omega\right\}d\xi,$$

provided that $\int_\alpha^\beta g(\omega)\cos\omega\xi d\omega$ approaches 0 with sufficient rapidity as $\xi\rightarrow\infty$; for example, if it is $O(\xi^{-\epsilon})$. Similar formulas for other ranges are also easy to obtain. The approximation found by the author is obtained by replacing $\exp\{\frac{1}{2}\int_\alpha^\beta g(\omega)\cos\omega\xi d\omega\}$ by

$$1+\tfrac{1}{2}\int_\alpha^\beta g(\omega)\cos\omega\xi d\omega+\frac{1}{2!}\left\{\tfrac{1}{2}\int_\alpha^\beta g(\omega)\cos\omega\xi d\omega\right\}^2.$$

M. Kac (Ithaca, N. Y.).

Feinberg, E. On the propagation of radio waves along an imperfect surface. Acad. Sci. USSR. J. Phys. **8**, 317–330 (1944). [MF 12549]

The author modifies the classical Sommerfeld formula for the field of a vertical dipole above a plane earth of high conductivity by introducing two different types of irregularity. For the first, the earth's surface, instead of being a

perfect plane, is defined by a function $\zeta(x, y)$ assuming that the gradients $\partial\zeta/\partial x$ and $\partial\zeta/\partial y$ are small throughout. For a uniformly and isotropically corrugated surface (one in which the deviations of the surface from an averaging plane $z=0$ are equal on both sides) it is shown that Sommerfeld's formula can be used to determine the average field by a suitable modification of the electrical constants. In the other type of irregularity the electrical properties of the earth are defined by a variable parameter $\eta(x, y)$, where η is the reciprocal of the complex dielectric constant and is assumed to be always small. In this case an integral equation for the vertical component of the electric field in the air is obtained and solved by a perturbation method starting from the Sommerfeld field. Both types of irregularity may be superposed and the author also indicates how deviations of the field from the average field may be obtained.

M. C. Gray (New York, N. Y.).

Feinberg, E. L. The propagation of radio waves along a real surface. Bull. Acad. Sci. URSS. Sér. Phys. [Izvestia Akad. Nauk SSSR] **8**, 109–131 (1944). (Russian) [MF 12472]

A modified translation is reviewed above. A continuation [same Bull. **8**, 200–209 (1944)] has already been quoted in these Rev. **6**, 166.

Fock, W. A. Diffraction of radio-waves around the globe. C. R. (Doklady) Acad. Sci. URSS (N.S.) **46**, 310–313 (1945). [MF 12951]

This paper is a brief account of investigations which are to be published as a monograph. No proofs are given. Notations: a=radius of the earth, h=height of transmitter above the surface of the earth, $k=2\pi$/wave-length, R=distance from transmitter to receiver, γ=angle between the vertical at the receiver and the direction to the transmitter, θ=angle subtended by transmitter and receiver at the center of the earth.

The field quantities at the surface of the earth can be represented in the form $S=\frac{1}{2}i\int \nu\varphi(\nu) \sec \nu\pi P_{\nu-\frac{1}{2}}(-\cos\theta)d\nu$, where $\varphi(\nu)$ is an expression containing Bessel functions and the contour of integration encloses the poles $\nu=\frac{1}{2}, \frac{3}{2}, \cdots$ of the integrand. An approximation to S is

$$S_1=\frac{e^{-i\pi/4}}{\pi(2\sin\theta)^{\frac{1}{2}}}\int \nu\varphi(\nu)e^{i\nu\theta}\frac{\Gamma(\nu+\frac{1}{2})\Gamma(\frac{1}{2})}{\Gamma(\nu+1)} \times F(\tfrac{1}{2}, \tfrac{1}{2}; \nu+1; -\tfrac{1}{2}ie^{i\theta}/\sin\theta)d\nu,$$

where the contour of integration encloses the poles (in the first quadrant) of $\varphi(\nu)$.

If $p=(\frac{1}{2}ka)^{\frac{1}{3}}\cos\gamma\gg1$, the Bessel functions in $\varphi(\nu)$ may be replaced by their Debye asymptotic representations; an approximate evaluation by means of the stationary phase method gives the "reflection formula" if $kh\cos\gamma\gg1$, and the formula of Weyl and van der Pol if $1\ll R^2/h^2\ll(ka)^{\frac{1}{3}}$ and $1\ll kR\ll a/h$. In the penumbral region, for which $|p|$ is of the order of unity, the asymptotic expressions involving Airy integrals must be used for the Bessel functions. The approximation S_1 can then be expressed as an integral in the integrand of which Airy integrals appear. This expression, which depends on two variables (and a parameter), is in its turn approximately reduced to an integral containing only one variable (and the parameter) and is then amenable to tabulation.

A. Erdélyi (Edinburgh).

Harrison, Charles W., Jr. Mutual and self-impedance for collinear antennas. Proc. I. R. E. **33**, 398–408 (1945).

This paper extends the Hallén integral equation method for determining the current distribution in an antenna to the case of two identical perfectly conducting collinear antennas, center-driven by equal generators either in phase or in opposite phase. Approximate solutions in inverse powers of the Hallén parameter $\Omega=2\log 2l/a$ are derived and curves are drawn of the mutual and self-impedance as functions of the spacing between the antennas for total lengths $\lambda/2$ and λ, using $\Omega=10$. These curves include only the first-order correction terms in $1/\Omega$ and are thus only of qualitative interest.

M. C. Gray (New York, N. Y.).

Harrison, Charles W., Jr. On the distribution of current along asymmetrical antennas. J. Appl. Phys. **16**, 402–408 (1945). [MF 12658]

This paper treats a problem similar to that reviewed above, since a grounded vertical radiator is essentially equivalent to a collinear pair, the difference being that the antenna and its image are now assumed to be driven at an arbitrary point instead of center-driven. The same method is used and in this case only formal results are given. The author also discusses the problem of an isolated antenna with generator voltage applied at an arbitrary point.

M. C. Gray (New York, N. Y.).

Quantum Mechanics

Dirac, Paul A. M. Quantum electrodynamics. Communications Dublin Inst. Advanced Studies. Ser. A. [Sgrîbh. Inst. Árd-Léigh. Bhaile Átha Cliath] no. 1, 36 pp. (1943). [MF 12887]

This paper summarizes the work of the author on a method, which avoids the convergence difficulties of quantized field theories, for handling problems concerned with the interaction of a number of charged particles with an electromagnetic field. An analysis and applications of this method have been given by Pauli [Rev. Modern Phys. **15**, 175–207 (1943)]. There are two distinct steps involved, a classical one and a quantum mechanical one. The first is the elimination of the infinite self-energy of the electron by modifying the vector potential entering into the Hamiltonian in accordance with the method proposed by the author [Proc. Roy. Soc. London. Ser. A. **167**, 148–169 (1938)] and involving the "λ-limiting" process. This assumes that there exists a time-like vector λ^s such that, if Z_i^s and Z_j^s are the coordinates of the ith and jth particles, then $Z_i^s-Z_j^s\pm\lambda^s$ is a space-like vector. Necessarily, then, the particles do not come closer together than the distance measured by the length of the vector λ. The limit as $\lambda\rightarrow0$ is then taken. This process can be taken over into quantum mechanics by suitably modifying the commutation relations for the vector potentials describing the electromagnetic field. The quantum mechanical modification of the usual treatment of quantized field theories involves the use of an indefinite Hermitian metric in the Hilbert space of the wave functions describing the states of the system. For spin-less particles the Gordon-Klein current vector is used. As a result both negative energies and negative probabilities occur.

For the system composed of n electrons and a radiation field it is assumed that initially no photons of positive or

negative energy are present. At a later time the wave function describing the system will depend on the variables referring to photons of both positive and negative energy. These are to be reinterpretated as follows. "The various parts of the wave function which referred to the existence of positive and negative energy photons in the old interpretation now refer to the emissions and absorptions of photons." By using Einstein's laws of radiation it is shown, for a hypothetical mathematical world with an initial probability distribution containing the positive and negative probabilities which occur in the theory, that the probability of an absorption process is negative and that of an emission process is positive. One then obtains positive values for the probability coefficients, the probabilities of a transition per unit time per unit intensity of radiation. These are assumed to be the same for the actual world.

The paper closes with the following statement. "The theory involves processes in which a certain photon is emitted and the same photon is absorbed—two actions which cancel each other and leave nothing observable. However, according to Einstein's laws, such processes would be stimulated by incident radiation in a different way from what they would be if these actions did not occur, and thus there is a possibility of getting experimental evidence for such actions." *A. H. Taub* (Princeton, N. J.).

Schönberg, M. M. Règles relativistes de commutation dans la théorie quantique des champs. J. Phys. Radium (8) 1, 201–209 (1940). [MF 12891]

In part I of this paper, the author considers the relativistic commutation rules of the electromagnetic potentials [P. Jordan and W. Pauli, Z. Physik **47**, 151–173 (1928)]:

$$[A_i(\mathbf{r}, t), A_{i'}(\mathbf{r}', t')] = \pm i\hbar\Delta(\mathbf{r}-\mathbf{r}', t-t')\delta_{ii'},$$
$$\Delta(\mathbf{r}, t) = r^{-1}(\delta(r+t)-\delta(r-t)), \qquad c=1,$$

and their analogue for the wave function ψ of an electron:

$$\psi^{(\rho)}(\mathbf{r}, t, k)\psi^{(\sigma)}(\mathbf{r}', t', k')+\psi^{(\sigma)}(\mathbf{r}', t', k')\psi^{(\rho)}(\mathbf{r}, t, k) = (1-\delta_{\sigma\rho})(\mathbf{r}tk|B|\mathbf{r}'t'k')$$

($\sigma, \rho = 1, 1; 1, 2;$ or $2, 2$), where $\psi^{(1)}=\psi$, $\psi^{(2)}=\psi^*$ and k is the spin coordinate. The $(\mathbf{r}tk|B|\mathbf{r}'t'k')$ are elements of Dirac's relativistic density matrix R_F (R_F is the solution of Dirac's equation which reduces to $\delta(\mathbf{r}-\mathbf{r}')\delta_{kk'}$ when $t=t'$). By means of the functions Δ and B, the integration of d'Alembert's and Dirac's equations is effected; the Green's functions (vanishing for $t=t'$) of these equations are easily obtained in terms of Δ and B.

This procedure is generalized in part II to a system with arbitrary first or second order linear equations of motion which are derived from a variational principle. The results are applied to Maxwell's equations and the relativistic commutation rules of the electric and magnetic field intensities are derived. *A. Schild* (Toronto, Ont.).

Beck, Guido. Remarque sur la notion du champ électromagnétique dans la théorie de Dirac. Portugaliae Phys. 1, 93–94 (1944). [MF 12937]

Rosen, Nathan. On waves and particles. J. Elisha Mitchell Sci. Soc. 61, 67–73 (1945). [MF 12988]

Arnous, Edmond. Utilisation de la fonction caractéristique de Laplace en mécanique ondulatoire, pour condenser le principe des valeurs propres et le principe de décomposition spectrale en un principe unique de quantification. C. R. Acad. Sci. Paris **218**, 108–109 (1944). [MF 13459]

Arnous, Edmond. Utilisation de la fonction caractéristique de Laplace dans certains problèmes classiques de mécanique ondulatoire. C. R. Acad. Sci. Paris **218**, 141–142 (1944). [MF 13463]

Destouches, Jean-Louis. Le rôle des transformations de Lorentz en mécanique ondulatoire et l'interprétation physique de la mécanique relativiste des systèmes de corpuscules. C. R. Acad. Sci. Paris **218**, 642–644 (1944). [MF 13474]

Eliezer, C. Jayaratnam, and Mailvaganam, A. W. On the classical theory of radiating electrons. Proc. Cambridge Philos. Soc. **41**, 184–186 (1945). [MF 12850]

This is a contribution to Dirac's classical theory of radiating electrons in which Lorentz's equations of motion with radiation damping,

$$m\dot{v}_\mu - \tfrac{2}{3}e^2(\ddot{v}_\mu+\dot{v}^2 v_\mu) = ev_\nu F_\mu{}^\nu, \qquad \mu=0, 1, 2, 3,$$

are assumed to hold exactly [Proc. Roy. Soc. London. Ser. A. **167**, 148–169 (1938)]. In this theory only a part of the mathematically possible solutions seem to be physically admissible within the limits of classical physics, and the question arises of finding an additional principle which singles out the physically admissible solutions. The authors propose such a principle, which they call the principle of radiative equilibrium and which reads $\ddot{v}_\mu+\dot{v}^2 v_\mu \to 0$ as $s\to\infty$, s being the proper time. It means that in the course of time the equations of motion tend to the Lorentz equations without radiation damping, $m\dot{v}_\mu = ev_\nu F_\mu{}^\nu$. The example of an electron in a uniform electric field is worked out and other examples are quoted [Eliezer, Proc. Indian Acad. Sci., Sect. A. **21**, 31–33 (1945)], showing that the authors' additional principle leads to good agreement with what one would expect on physical grounds. *P. Weiss* (London).

Tonnelat, Marie-Antoinette. Sur la théorie du photon dans un espace de Riemann. Ann. Physique (11) **15**, 144–224 (1941). [MF 12878]

De Broglie's theory [Actualités Sci. Ind., nos. **181** (1934), **411** (1936)], in which the photon is treated as a particle which satisfies Dirac's relativistic wave equation and its adjoint equation simultaneously, is extended from a Euclidean to a Riemannian space-time. In the author's presentation the "wave-function" of the photon is a 4×4 matrix which is the irreducible representation of the full 16-component Dirac-Eddington algebra,

$$\psi = \gamma^0\phi_0+\gamma^\mu\phi_\mu+\tfrac{1}{2}\gamma^\mu\gamma^\nu\phi_{\mu\nu}+\tfrac{1}{6}\gamma^\mu\gamma^\nu\gamma^\rho\phi_{\mu\nu\rho}+\tfrac{1}{24}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\phi_{\mu\nu\rho\sigma},$$

where the ϕ_0, ϕ_μ, $\phi_{\mu\nu}$, etc., behave as scalars, vectors, tensors, etc. The wave equations are obtained by operating on ψ with $\gamma^\mu\partial/\partial x^\mu - k$ from the left and with $\gamma^\mu\partial/\partial x^\mu + k$ from the right, where $k=(2\pi/h)\mu_0 c$. Thus de Broglie's theory is expressed in terms of tensors instead of in terms of spinors. According to a theorem by É. Cartan [Actualités Sci. Ind., no. **701** (1938), pp. 89–91], this fact is essential for an extension to a Riemannian space to be possible.

The extension is carried out on the lines of Schroedinger's earlier attempt to extend Dirac's equation [S.-B. Preuss. Akad. Wiss. **1932**, no. 11–12 (1932)]. The $\gamma^\mu(x)$ satisfy $\tfrac{1}{2}(\gamma^\mu\gamma^\nu+\gamma^\nu\gamma^\mu)=g^{\mu\nu}$ and, as regards the covariant differentiation, besides the Riemann-Christoffel symbol $\Gamma^\mu_{\rho\sigma}$ a matrix vector Γ_μ appears which connects $\gamma^\mu(x)$ with $\gamma^\mu(x+dx)$. The relation

$$\gamma^\mu{}_{:\rho} \equiv \partial\gamma^\mu/\partial x^\rho + \Gamma^\mu_{\sigma\rho}\gamma^\sigma - (\Gamma_\rho\gamma^\mu - \gamma^\mu\Gamma_\rho) = 0$$

determines the Γ_μ. Various equivalent forms of writing the

wave equations are given and their compatibility conditions discussed. It is found that these are always satisfied if the space has no torsion.

The electromagnetic equations are derived and their solution corresponding to the annihilation of a photon is given. The second order electromagnetic equations suggest a relation between the mass μ_0 of the photon and the radius of the universe. The whole theory applies to any particle of spin one. *P. Weiss* (London).

Tonnelat, Marie-Antoinette. La particule de spin 2 et la loi de gravitation d'Einstein dans le cas de présence de matière. C. R. Acad. Sci. Paris **218**, 305–308 (1944). [MF 13395]

The author's theory of the "graviton," according to which the wave equation of particles with maximum spin two comprises the linear approximation of Einstein's gravitational theory, is extended to include the interaction with matter. A combined wave function χ for the graviton and the material particle is set up which satisfies the wave equation

$$(h/2\pi i)\sum_{\nu=1}^{4}\beta_4^{(\nu)}\partial\chi/\partial t=H_D\sum\beta_4^{(\nu)}+\mathfrak{H}_G\chi+H^{(1)}\chi.$$

Here $\beta_4^{(\nu)}$ denotes a product of Dirac matrices, H_D is the Hamiltonian of Dirac's theory, $\mathfrak{H}_G$ that of the graviton, $H^{(1)}$ a perturbing Hamiltonian describing the interaction; $H^{(1)}$ is expressed in terms of basic quantities of the theory. Nearly but not quite complete agreement with Einstein's gravitational equations in the presence of matter is obtained. The agreement is perfected by the introduction of a further Hamiltonian $H^{(2)}$ arising from the interaction due to a state of spin zero. *P. Weiss* (London).

Tonnelat, Marie-Antoinette. Sur l'interaction entre deux particules matérielles au moyen du corpuscule de spin maximum 2; loi de gravitation newtonienne. C. R. Acad. Sci. Paris **218**, 139–141 (1944). [MF 13462]

Petiau, Gérard. Sur l'équation d'ondes du corpuscule de spin total maximum $2(h/2\pi)$. Revue Sci. (Rev. Rose Illus.) **81**, 499–500 (1943). [MF 13803]

Petiau, Gérard. Sur la représentation des interactions massiques par l'intermédiaire des ondes longitudinales de la théorie de la particule de spin $2(h/2\pi)$. C. R. Acad. Sci. Paris **218**, 34–36 (1944). [MF 13449]

Petiau, Gérard. Sur les équations d'ondes macroscopiques du corpuscule de spin 2 en présence de matière. C. R. Acad. Sci. Paris **218**, 136–138 (1944). [MF 13476]

Petiau, Gérard. Sur les interactions entre particules matérielles s'exerçant par l'intermédiaire de la particule de spin $2(h/2\pi)$. J. Phys. Radium (8) **6**, 115–120 (1945). [MF 13793]

Kwal, Bernard. Sur la mécanique ondulatoire des corpuscules élémentaires. C. R. Acad. Sci. Paris **218**, 613–615 (1944). [MF 13470]

Continuation of a paper in the same C. R. **218**, 548–550 (1944); these Rev. **6**, 224.

Schönberg, Mario. The "self-energy" of the electron. Anais Acad. Brasil. Ci. **17**, 163–165 (1945). (Portuguese) [MF 13369]

Walsh, Pius. The point singularity in a non-linear meson theory. Proc. Roy. Irish Acad. Sect. A. **50**, 167–187 (1945). [MF 12689]

In Schrödinger's "unitary field theory" [same Proc. **49**, 225–235 (1944); these Rev. **6**, 72] the electromagnetic field equations differ from Maxwell's in two respects: (1) Born's hypothetical nonlinear modification and (2) Proca's self-exiting term which corresponds to a nonvanishing rest mass. For the electromagnetic field which Schrödinger treated, the rest mass in question is that of the photon, which vanishes or at least is extremely small, and there is no overlapping of the effects of the two complications. Modification (1) affects only the immediate neighborhood of the particle while (2) affects only regions at a large distance from it. For the meson field, however, there is considerable overlapping of the two effects. Both (1) and (2) come into play in the immediate neighborhood of the particle. The present paper attacks this problem and a qualitative picture of the situation is obtained. The field equations are solved first for the one-dimensional case and then for the spherically symmetric static case, in the two limiting approximations $\mu\sqrt{g}\ll 1$ and $\mu\sqrt{g}\gg 1$, where g is the meson charge and μ is the rest mass of the meson. The main results obtained are the expressions for the total field energy of the meson in terms of μ and g in these approximate cases.
S. Kusaka (Northampton, Mass.).

Heitler, W. On the production of mesons by proton-proton collisions. II. Proc. Roy. Irish Acad. Sect. A. **50**, 155–165 (1945). [MF 12690]

This paper deals with revisions and modifications of the previous calculations made in part I by the author and Peng [same Proc. **49**, 101–133 (1943); these Rev. **5**, 166] on the production of mesons by proton-proton collisions. In the approximate method of Weizsäcker and Williams used in the calculations, the collision between a high energy proton and another at rest is treated as equivalent to a beam of virtual mesons passing by a stationary proton. The mesons which are scattered in the latter process correspond to those which are created in the former. Due to this approximation, the mesons produced can be considered as being made up of (1) those emitted by the nucleon at rest due to the disturbance which it receives from the passing fast nucleon and (2) those arising from the disturbance which the fast nucleon receives from the stationary nucleon. In the original paper the authors tried to include the second contribution by a mathematical device which now seems incorrect, and the purpose of the present paper is to improve this method and work out the modifications. The new procedure is to calculate the second contribution by making a Lorentz transformation to a system in which the fast nucleon is transformed to rest and the rest particle moves in the opposite direction. The new result differs from the old in that though the two contributions yield the same total number of mesons, the energy spectrum of the mesons emitted by the second contribution is very different from that due to the first and much more in favor of high meson energies. The net result is that the energy loss of the proton is not very different from the previous result up to about $5Mc^2$ for the proton energy but is much greater for higher energies. *S. Kusaka* (Northampton, Mass.).

Hulthén, Lamek. Comments on the difficulties of the meson theory. Rev. Modern Phys. **17**, 263–266 (1945). [MF 13694]

Shaffer, Wave H. Degenerate modes of vibration and perturbations in polyatomic molecules. Rev. Modern Phys. 16, 245–259 (1944). [MF 12534]

Kiang, A. T. Vibrational-rotational spectrum and potential function of a linear asymmetrical triatomic molecule. Chinese J. Phys. 5, 49–63 (1944). [MF 13760]

Wang, J. S., and Mei, Jenn-Yueh. On the application of Kirkwood's theory of order-disorder transformation to adsorption. Chinese J. Phys. 5, 64–88 (1944). (English. Chinese summary) [MF 13761]

Auluck, F. C., and Kothari, D. S. The energy levels of 'holes' in liquids. Proc. Cambridge Philos. Soc. 41, 180–183 (1945). [MF 12849]

Klein, O., and Lindhard, J. Some remarks on the quantum theory of the superconductive state. Rev. Modern Phys. 17, 305–309 (1945). [MF 13696]

Lifshitz, I. M. Optical behavior of non-ideal crystal lattices in the infra-red. III. Acad. Sci. USSR. J. Phys. 8, 89–105 (1944). [MF 12338]

[Parts I and II appeared in the same J. 7, 215–228, 249–261 (1943); these Rev. 6, 112, 221.] Crystals containing a few foreign atoms are studied when the nonvanishing elements of the perturbation matrix Λ^* are not small, while the chance that they differ from zero is small. The concentration c_s of the admixture atoms thus being small, an expansion in powers of c_s is used for the mean moment **P**. The linear terms are considered first and are expressed in terms of the additional moment **p** due to a perturbing center located in the sth place in one of the lattice cells. The sum of products $c_s\mathbf{p}_s$ is multiplied by a factor N which is the inverse of the smallest possible variation of the concentration c_s. The quantity $\mathbf{p}_s$ is found by writing down the vibration equation with perturbations. The perturbation matrix is expressed as an infinite determinant. Possible simplifications are considered and the equation to be solved exactly is written in matrix form. For sufficiently large values of the parameter z (or ω^2) in the equation $(A-z+\Lambda)u=\varphi+\delta$, an expansion is used involving repetitions of the operator $C\varphi=(A-z)^{-1}\varphi$. In fact, u is put in the form $u=u_1+u_2$, where u_1 is derived from φ, u_2 from δ, and $u_1=u_0-C\Lambda u_0+C\Lambda C\Lambda u_0-C\Lambda C\Lambda C\Lambda u_0+\cdots$, where u_0 is a solution of the unperturbed equation. A transition to an arbitrary z is made after the solution is written in closed form. The final expressions involve both summations and integrals. For values of z lying in the interval of eigenfrequencies the imaginary parts of some of the integrals lead to absorption; absorption in mixed crystals is considered first. The appearance of new resonance frequencies by the splitting up of each of the frequencies z_r into p frequencies $z_{r,s}$ is noted. This corresponds, in this approximation, to the presence of one atom of the admixture, which could be located in any of the p positions in the cell. A reference is made to confirming experimental results of Kruger, Reinkorber and Koch-Holm. The expansion in powers of the concentration is carried further as further terms would reveal the existence of new resonance frequencies which remain isolated. In the appendices a few questions connected with averaging are investigated and the main disadvantage of the present method is discussed.

H. Bateman (Pasadena, Calif.).

Thermodynamics, Statistical Mechanics

Cowling, T. G. The electrical conductivity of an ionized gas in a magnetic field, with applications to the solar atmosphere and the ionosphere. Proc. Roy. Soc. London. Ser. A. 183, 453–479 (1945). [MF 12694]

The author begins with a summary of earlier results given in the book of Chapman and Cowling [The Mathematical Theory of Non-uniform Gases, Cambridge, 1939; these Rev. 1, 187] and then shows that the discrepancy between values of the relative velocity of diffusion of the two constituents of a binary mixture obtained by the free-path theory and the method of velocity distribution arises from the assumption in the free-path theory that a molecule, directly after collision, has, on the average, no peculiar velocity. This assumption is invalid for unlike molecules and the necessary modification deprives the method of the free-path of its chief simplicity. The method used here is based on the distribution of velocities.

The results of the first approximation for a binary gas are extended to a multiple gas, expressions being found for the velocity of diffusion $\bar{C}_r$ of each constituent and for the total current-density j. In the presence of a magnetic field H an electric field E produces two components of current, one parallel to E, the other perpendicular to both E and H. The respective coefficients entering into the expression for j are called direct and transverse conductivities. The analysis is carried further and certain quantities are regarded as generalized diffusion coefficients for direct and transverse diffusion. Expressions are found also for the direct and transverse conductivities. The case of a ternary mixture is worked out in detail for this first approximation both when the gas is slightly ionized and when electrons are present.

In the solar applications a spot field of the order of 2000 gauss produces marked changes in the conductivity in the surface layers. The electron conductivity is of chief interest and is studied with the aid of a mean collision distance for molecules of the rth and sth gases. When collisions with neutral atoms are ignored a general formula for this conductivity may be found. Some numerical results are given. A consideration of mechanical effects leads to some interesting remarks about the surface layers near sunspots and the reason why prominences often end their existence by being sucked into a spot.

In the application to the earth's upper atmosphere estimates of conductivity are made for the E and F layers and some of the results are illustrated by a graph. Brief discussions of Chapman's dynamo theory of the lunar variations of the earth's magnetic field, of diamagnetism and drift currents conclude this section. It is stated that the theory is not yet adequate to explain all the phenomena.

In an appendix reasons are given to justify the use of Boltzmann's equation for ionized gases even though after a short collision between a molecule A and a molecule B the forces of interaction between A and molecules other than B are at least comparable with the force exerted by the magnetic field in all situations. It is thought that for every electrostatic encounter producing a large deflection there are many producing small deflections and that these are in various directions and cancel out on the average while the effect of the magnetic field is cumulative. Thus in a sunspot the magnetic forces dominate the interaction forces. Motions in the outer layers of the sun are directed almost exactly along the lines of magnetic force, while in a sunspot conditions of "free spiraling" are approached.

H. Bateman (Pasadena, Calif.).

Dufour, Louis. Sur les variations bariques et thermiques de l'atmosphère. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **29**, 374–380 (1943). [MF 13850]

Van Mieghem, Jacques. Sur les transformations adiabatiques et isobariques de l'air atmosphérique humide. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **29**, 608–620 (1943). [MF 13857]

Vlasov, A. On the kinetic theory of an assembly of particles with collective interaction. Acad. Sci. USSR. J. Phys. **9**, 25–40 (1945). [MF 13374]

This is an almost literal translation of the Russian version [Bull. Acad. Sci. URSS. Sér. Phys. [Izvestia Akad. Nauk SSSR] **8**, 248–266 (1944); these Rev. **6**, 222, 334]. Preceding the chapter dealing with spontaneous formation of crystal structure, a chapter is added which is concerned with nonstationary solutions of the initial "wave" equation.

H. G. Baerwald (Cleveland Heights, Ohio).

Gabor, D. Stationary electron swarms in electromagnetic fields. Proc. Roy. Soc. London. Ser. A. **183**, 436–453 (1945). [MF 12693]

In the theory of the simple magnetron, as well as in the study of the correction of electron-optical systems by means of space charges, it is important to have a theory of the mechanics of clouds of electrons rotating in axially symmetrical magnetic fields. Beyond unsatisfactory rudiments, no such theory exists. The great difficulty appears to lie in the mutual interaction of the electrons. It is the object of this paper to take a preliminary step toward such a theory by furnishing a comprehensive discussion of electron clouds in the absence of interaction. The general method is to treat the problem as one of statistical mechanics. The first step is to obtain the distribution law on the basis of classical statistical mechanics: the phase space of a one-electron problem under a fixed axially symmetrical electromagnetic potential is studied, and uniform distribution therein is assumed. [The reviewer objects to the argument by the ergodic theorem purporting to establish the uniformity of the distribution, since there is a singularity in the flow, corresponding to passage of stream lines through the cathode, and metrical transitivity appears to be in default; he does not regard this a serious matter, since there are certainly many good physical reasons for assuming the uniformity of the distribution.] The second step is to substitute the corresponding space density, corresponding to the uniform phase-space density, into Poisson's equation in order to calculate the electrostatic potential which would be produced by a cloud of electrons of space density proportional to the space (probability) density just obtained for the single electron. From this point on the method of the self-consistent field is used. Only tentative conclusions are drawn, particularly in a comparison with previous theories.

B. O. Koopman (Washington, D. C.).

Yarnold, G. D. The energies of uniformly accelerated particles in a gas. Philos. Mag. (7) **36**, 185–200 (1945). [MF 13342]

Physical considerations and idealizations lead to the following problem. In the space between two infinite planes (electrodes) resting spheres (slowly moving molecules) are distributed at random. Uniformly accelerated electrons collide with them and lose energy according to the laws of elastic collision. The author computes the mean energy of the electrons.

W. Feller (Ithaca, N. Y.).

De Donder, Th. Les micromodèles et les macromodèles dans la théorie nouvelle de la mécanique statistique. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **27**, 689–696 (1941). [MF 12679]

De Donder, Th. Les micromodèles et les macromodèles dans la théorie nouvelle de la mécanique statistique. II. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **29**, 63–70 (1943). [MF 13838]

San Juan, R. On the finiteness of the entropy and the vanishing of the specific heat in approach to absolute zero. Revista Acad. Ci. Madrid **36**, 415–417 (1942). (Spanish) [MF 12782]

Proposition P: the specific heat C of every substance approaches 0 as the absolute temperature $T \to 0$. The author points out that P does not follow from the principle of Nernst alone (as is often supposed), but that P does follow from this principle and the (apparently true) proposition that C decreases with T as $T \to 0$.

C. C. Torrance.

San Juan, Ricardo. An application of abstract spaces to the phase law of thermodynamics. Las Ciencias. Madrid **8**, no. 2, 2 pp. (1943). (Spanish) [MF 12784]

Sábato, Ernesto. The concept of temperature in phenomenological thermodynamics. Revista Union Mat. Argentina **10**, 109–127 (1945). (Spanish) [MF 12505]

Verschaffelt, J. E. La thermomécanique des processus irréversibles. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **28**, 490–495 (1942). [MF 13668]

Comments on a paper of Eckart [Phys. Rev. (2) **58**, 267–275 (1940)].

C. C. Torrance (Cleveland, Ohio).

Verschaffelt, J.-E. La thermomécanique des fluides en mouvement stationnaire et l'effet Joule-Thomson. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **26**, 193–210 (1940). [MF 13831]

The author develops general equations for steady fluid flow corresponding to the first and second laws of thermodynamics. These are obtained by dynamical reasoning and include friction, external heat sources and energy changes through chemical processes. The results are applied to an analysis of the Joule-Thomson effect.

N. A. Hall.

BIBLIOGRAPHICAL NOTES

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